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Discrete Shocks for Difference Approximations to Systems of Conservation Laws*

DANIEL MICHELSON

University of California, Los Angeles, California 90024

The existence of weak discrete shocks for a wide class of difference approximations to systems of conservation laws is proved. The difference schemes have to be conservative, kth order accurate, and, roughly speaking, $(k+1)$th order dissipative, where $k = 1$ or 3. The proof makes use of the central manifold theorem for an implicit map and of the fact that the stable and unstable manifolds of the differential flow $y^{(k)} = y^2 - 1$ for $k = 3$ intersect transversally.

1. INTRODUCTION

In this paper we study the planar discrete shocks for difference schemes approximating the system of conservation laws

$$\sum_{i=1}^{m} (f_i(u))_{x_i} = 0, \tag{1.1}$$

where $f_i : R^n \to R^n$ are smooth vector functions, and the unknown function $u = u(x) \in R^n$ depends on the vector variable $x = (x_1, x_2, \ldots, x_m) \in R^m$. One can view the problem in (1.1) as a stationary or as an evolutionary one. In the last case one can assume that x_m is the time variable and correspondingly $f_m(u) = u$. By a planar shock solution to system (1.1) we mean the solution

$$\begin{aligned} u(x) &= u_L, \qquad \langle x, s\rangle < 0 \\ &= u_R, \qquad \langle x, s\rangle > 0, \end{aligned} \tag{1.2}$$

where $s = (s_1, s_2, \ldots, s_m) \in R^m$ is a constant vector, and the states u_L and u_R satisfy the Rankine–Hugoniot condition

$$f(u_L) = f(u_R), \qquad \text{where} \quad f(u) = \sum_{i=1}^{m} s_i f_i(u). \tag{1.3}$$

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We shall keep the vector s *fixed* while u_L and (the corresponding) u_R vary in a neighborhood of some compact $(n-1)$-dimensional manifold (with boundary) $S \subset R^n$. We assume that the differential $df[u_0]$ of the function f at any $u_0 \in S$ has a *simple zero* eigenvalue $\lambda_0(u_0) = 0$ with the corresponding right and left eigenvectors $r_0(u_0)$ and $l_0(u_0)$. The eigenvalue $\lambda_0(u_0)$ also has to be genuinely nonlinear on S; i.e., the directional derivative

$$d\lambda_0[u_0](r_0(u_0)) \neq 0 \qquad \text{for any} \quad u_0 \in S. \tag{1.4}$$

Then, as in [5, pp. 479–480] one can show that Eq. (1.3) considered in a small neighborhood of S has a unique solution $u_R = \varphi(u_L)$. The function φ is smooth and for any $u_0 \in S$ satisfies the conditions

$$\varphi(u_0) = u_0 \tag{1.5}$$

and

$$d\varphi[u_0](r_0(u_0)) = -r_0(u_0). \tag{1.6}$$

The shock in (1.2) is called entropy satisfying if

$$\lambda_0(u_L) > 0 > \lambda_0(u_R) \tag{1.7}$$

and entropy violating if

$$\lambda_0(u_L) < 0 < \lambda_0(u_R). \tag{1.8}$$

According to (1.4)–(1.6) the pair $u_L, u_R = \varphi(u_L)$ with u_L close to S provides an entropy satisfying shock if $\lambda_0(u_L) > 0$, and an entropy violating one if $\lambda_0(u_L) < 0$. We approximate system (1.1) by a difference scheme

$$G(\{E_x^j u(x)\}) = 0, \qquad j = (j_1, j_2, \ldots, j_m) \in J \subset \mathbf{Z}^m. \tag{1.9}$$

Here J is a finite set, $G = G(\{u_j\})$ is a smooth[1] vector function of vector variables $u_j \in R^n$, $j \in J$, $E_x^j = E_{x_1}^{j_1} \cdot E_{x_2}^{j_2} \cdot \ldots \cdot E_{x_m}^{j_m}$, and E_{x_i}, $i = 1, 2, \ldots, m$, is the shift operator

$$E_{x_i} u(x_1, \ldots, x_i, \ldots, x_m) = u(x_1, \ldots, x_i + h, \ldots, x_m). \tag{1.10}$$

We consider $u(x)$ in (1.9) as a grid function defined on a uniform grid R_h^m with the mesh size h. (The uniformity of the grid is not a restriction since one can rescale the x_i coordinates in (1.1).) We assume that the difference scheme in (1.9) is conservative; i.e.,

$$G(\{E_x^j u(x)\}) = \sum_{i=1}^{m} (E_{x_i} - I) G_i(\{E_x^j u(x)\}) \tag{1.11}$$

[1] Here and elsewhere by smooth we mean C^r smoothness with $r \geq 3$.

with multi-index j in G_i varying over corresponding subsets of J. The scheme should be also consistent with the system in (1.1), which is to say

$$G_i(\{u\}) = f_i(u), \qquad i = 1, 2, \ldots, m, \tag{1.12}$$

where by $\{u\}$ we denote the set $\{u_j\}$ of vectors $u_j = u$. Let the components s_i of the vector s be *rational* numbers with a least common denominator q:

$$s_i = p_i/q, \qquad q > 0, \qquad i = 1, 2, \ldots, m. \tag{1.13}$$

DEFINITION. The grid function $u(\tau)$, $\tau \in R^1_{h/q}$, is called a discrete shock solution of the difference scheme (1.9) in the direction of the vector s, if the function $u(\langle x, s\rangle)$, $x \in R^m_h$ is a solution of (1.9) and

$$\lim_{\tau \to -\infty} u(\tau) = u_L, \qquad \lim_{\tau \to +\infty} u(\tau) = u_R. \tag{1.14}$$

The states u_L and u_R necessarily satisfy the Rankine–Hugoniot condition (1.3). Obviously, $u(\tau)$ is a solution of the difference equation

$$G_\tau(\{E_\tau^{j_0} u(\tau)\}) = q^{-1} G(\{E_\tau^{\langle j, p\rangle} u(\tau)\}) = 0. \tag{1.15}$$

Here p is the integer vector $(p_1, p_2, \ldots, p_m)$, E_τ is the shift operator

$$E_\tau u(\tau) = u(\tau + h/q), \tag{1.16}$$

and the exponent j_0 varies over a finite set in $\mathbf{Z}$. From the conservation form (1.11) we deduce

$$\begin{aligned} G_\tau(\{E_\tau^{j_0} u(\tau)\}) &= q^{-1} \sum_{i=1}^{m} (E_\tau^{p_i} - I) G_i(\{E_\tau^{\langle j, p\rangle} u(\tau)\}) \\ &= (E_\tau - I) G_0(\{E_\tau^{j_0} u(\tau)\}). \end{aligned} \tag{1.17}$$

The consistency as in (1.12) implies

$$G_0(\{u\}) = f(u). \tag{1.18}$$

Therefore Eq. (1.15) with the conditions on infinity (1.14) is equivalent to

$$G_0(\{E_\tau^{j_0} u(\tau)\}) - f(u_L) = 0, \qquad j_0 \in J_0 \subset \mathbf{Z}, \tag{1.19}$$

so that indeed $f(u_L) = f(u_R)$. Our most restrictive assumption relates to the linearization of the difference operator G_τ at a constant solution $u(\tau) \equiv u_0 \in S$. Roughly speaking, for any $u_0 \in S$ we require the linearized operator to be the kth order accurate and $(k+1)$th order dissipative approximation of the differential operator $df[u_0] \cdot \partial/\partial\tau$. The number k

should be *odd*; however, we have been able to prove our main result only for $k = 1$ or $k = 3$.[2] The precise statement of the last assumption is given in the next section. Under the above assumptions we prove the following.

MAIN THEOREM. *For any u_L which belongs to a small one-sided neighborhood of the manifold S there exists a one parameter family of discrete shocks depending smoothly on u_L. For $k = 1$ this family is locally unique. The states u_L and u_R in* (1.14) *satisfy the entropy condition* (1.7) *if the dissipation of the scheme is positive* (*as in* (2.13)) *or violate this condition if the dissipation is negative.*

In the next section we state the Main Theorem in more detail including the asymptotic form of the discrete shocks modulo an error of order $|u_L - u_R|^2$.

Discrete shocks have been studied previously by Jennings [2] for scalar conservation laws and monotone schemes, and more recently by Majda and Ralston in [5] for systems of conservation laws approximated by *first* order accurate schemes. Our work could be considered as a generalization of [5]. Our accuracy and dissipativity assumptions for $k = 1$ coincide with the corresponding assumptions in [5]. However, we remove one significant restriction in [5] which implies that the operator G_0 in (1.19) has a non-zero Jacobian with respect to the most left and most right points u_{j_0} in the domain of dependence J_0. This restriction excludes, for example, the so-called upwind difference schemes. We also describe the form of the shock profile. Our proof is more simple than in [5] since we use the integrated equation (1.19) instead of the original (1.15). The results for $k = 3$ are new. We have been inspired by the important work of Kopell and Howard [3]. Their ideas of reduction to the central manifold and subsequent rescaling of the equations are adequate also in the difference case. As a result of rescaling we arrive at a two point difference system in R^k which could be interpreted as a difference approximation of the canonical differential system for a vector function

$$y = \left(y^{(1)}, y^{(2)}, \dots, y^{(k)}\right)^T$$

$$\frac{dy}{d\tau} = \left(y^{(2)}, y^{(3)}, \dots, y^{(k-1)}, g\left(y^{(1)}\right)\right)^T, \tag{1.20}$$

where $g\,(y^{(1)})) = \frac{1}{2}[(y^{(1)})^2 - 1]$. The points $y_L = (-1, 0, \dots, 0)^T$ and $y_R = (1, 0, \dots, 0)^T$ are critical points of the system. The parameter $\varepsilon \sim q^{-1}|\lambda_0(u_L)|^{1/k}$ plays in our approximation the role of a step size in τ. Kopell and Howard [3] proved for $k = 3$ that the two-dimensional stable and unstable manifolds of the points y_L and y_R correspondingly intersect

[2] *Note added in proof*: Using the ideas of Conley's index we are now able to prove the existence of weak discrete shocks (but not a one-parameter family) for *any* odd k.

transversally along an integral line connecting y_L and y_R. We use this result and order to show that for small ε the corresponding invariant manifolds of the difference approximation also intersect transversally. It is quite plausible that the same transversality holds for the system in (1.20) when k is any *odd* number. If so, the weak discrete shocks exist for difference schemes of any odd order of accuracy. For even k the system in (1.20) possesses a first integral, so that there is no integral line connecting y_L and y_R. For $k = 3$ the integral line connecting y_L and y_R is oscillating, which explains the so-called overshoot for higher order accurate schemes. Knowing the form of the shock profile could be also helpful in filtering out the numerical oscillations associated with the discrete shocks. In this work we assume the genuine non-linearity of the eigenvalue $\lambda_0(u_0)$. Recently, Pego [7] has considered dissipative differential perturbation of the system in (1.1) which, instead of genuine non-linearity, satisfies the strict entropy condition of Liu [4]. His differential problem corresponds to our difference one with $k = 1$. Using Pego's arguments one can show that our existence and uniqueness results for $k = 1$ and evolutionary problem (1.1) hold also in the case of the above entropy condition.

Let us outline the contents of this paper. In Section 2 we formulate and explain the accuracy and dissipativity conditions and state our Main Theorem. In Section 3 the difference problem (1.19) is reduced to a central manifold and then rescaled. In Section 4 the Main Theorem is proved. In Section 5 we prove the invariant manifold theorems for an implicit map. We also prove the intuitively obvious fact that the stable manifold of a one-step difference approximation to a differential system tends to a stable manifold of that system when the step size in the approximation tends to zero. Finally, in the Appendix we consider the system in (1.20) with $k = 3$ and generalize the transversality result of [3] to functions $g(y^{(1)})$ which are even and have a positive derivative for $y^{(1)} > 0$.

2. Preliminaries: The Main Theorem

We first describe the accuracy and dissipativity conditions for the reduced operator G_τ in (1.15). For any $u \in R^n$ define the matrices

$$C_j(u) = \left.\frac{\partial G(\{u_j\})}{\partial u_j}\right|\{u_j\} = \{u\}. \tag{2.1}$$

The difference operator

$$dG[u] = \sum C_j(u) E_x^j, \qquad j \in J \subset \mathbf{Z}^m \tag{2.2}$$

is called the linearization of the operator G in (1.9) and is essentially the differential of the operator G considered as a map in the space of mesh functions $u(x)$. The symbol of the linear operator $dG[u]$ is defined by

$$d\hat{G}[u](\xi) = \sum C_j(u) e^{i\langle \xi, j\rangle}, \qquad j \in J, \qquad \xi = (\xi_1, \xi_2, \dots, \xi_m) \in R^m. \tag{2.3}$$

Using the same definitions for the operator G_τ we get the identity

$$d\hat{G}_\tau[u](\xi_0) = q^{-1} d\hat{G}[u](p\xi_0), \qquad \xi_0 \in R^1. \tag{2.4}$$

ASSUMPTION 2.1. The difference operator $dG_\tau[u_0]$ for any $u_0 \in S$ is a kth order accurate approximation of the differential operator $df[u_0] \cdot \partial/\partial\tau$ in the direction of the eigenvector $r_0(u_0)$. In the terms of symbols it is to say

$$d\hat{G}_\tau[u_0](\xi_0) \cdot r_0(u_0) = i\xi_0 df[u_0](r_0(u_0)) + O(\xi_0^{k+1}) = O(\xi_0^{k+1}),$$
$$\text{for any} \quad u_0 \in S. \tag{2.5}$$

This assumption is obviously satisfied if the difference operator $dG[u_0]$ is a kth order approximation of the differential operator $\partial_u f_1(u_0)\partial/\partial x_1 + \cdots + \partial_u f_m(u_0)\partial/\partial x_m$, or in the terms of symbols,

$$d\hat{G}[u_0](\xi) = i(\partial_u f_1(u_0)\xi_1 + \cdots + \partial_u f_m(u_0)\xi_m) + O(|\xi|^{k+1}). \tag{2.6}$$

Next, we impose on the operator $dG_\tau[u_0]$ the regularity or dissipativity condition.

ASSUMPTION 2.2. The symbol $d\hat{G}_\tau[u_0](\xi_0)$ is non-singular for any $u_0 \in S$ and $\xi_0 \neq 2n\pi$; i.e.,

$$\det d\hat{G}_\tau[u_0](\xi_0) = \det q^{-1} d\hat{G}[u_0](p\xi_0) \neq 0,$$
$$\text{for any } u_0 \in S \text{ and } \xi_0 \neq 0 \bmod 2\pi. \tag{2.7}$$

Majda and Ralston in [5] call it the "non-resonance" condition. The name "regularity" fits more the stationary problems while "dissipativity condition" is more appropriate for evolutionary problems. Obviously, if $dG[u_0]$ is regular or dissipative in the sense

$$\det \hat{G}[u_0](\xi) \neq 0 \qquad \text{for any } u_0 \in S \text{ and} \qquad \xi \neq 0 \bmod 2\pi, \tag{2.8}$$

then the above assumption holds, since the integers $p_1, \dots, p_m$ are relatively prime.

Our last condition relates to the remainder $O(\xi_0^{k+1})$ in (2.5) and is analogous to condition (3.15) in [5].

ASSUMPTION 2.3. The approximation $dG_\tau[u_0]$ is *exactly* kth order accurate in the eigenspace corresponding to the $\lambda_0(u_0)$ eigenvalue; i.e.,

$$l_0(u_0)d\hat{G}_\tau[u_0](\xi_0)\cdot r_0(u_0) = b(u_0)q^k\cdot(i\xi_0)^{k+1} + O(\xi_0^{k+2}), \tag{2.9}$$

where $b(u_0) \neq 0$. (Note that $b(u_0)$ is real.)

We assume that the vectors $l_0(u_0)$ and $r_0(u)$ in (2.9) are normalized in the following way:

$$l_0(u_0)\cdot r_0(u_0) = 1 \qquad \text{and} \qquad d\lambda_0[u_0](r_0(u_0)) = 1 \qquad \text{for any } u_0 \in S. \tag{2.10}$$

This implies also

$$l_0(u_0)d^2f[u_0](r_0(u_0)) = (l_0(u_0)\cdot r_0(u_0))\cdot d\lambda_0[u_0](r_0(u_0)) = 1, \tag{2.11}$$

where $d^2f[u_0](r_0(u_0))$ is the second derivative of $f(u)$ at $u = u_0$ in the direction of the vector $r_0(u_0)$. Note that (2.9) is equivalent to

$$l_0(u_0)d\hat{G}[u_0](s\xi_0)\cdot r_0(u_0) = b(u_0)(i\xi_0)^{k+1} + O(\xi_0^{k+2}), \tag{2.12}$$

which explains the factor q^k in (2.9). For odd k the term $b(u_0)(i\xi_0)^{k+1}$ represents the dissipation of the scheme $dG_\tau[u_0]$ in the $\lambda_0(u_0)$ eigenspace. We call the dissipation positive or negative if, correspondingly,

$$i^{k+1}b(u_0) > 0, \qquad u_0 \in S \tag{2.13}$$

or

$$i^{k+1}b(u_0) < 0, \qquad u_0 \in S. \tag{2.14}$$

Suppose that Eq. (1.1) is evolutionary so that x_m plays the role of time and $f_m(u) = u$. The following result, which generalizes Proposition 2.3 in [5], relates the sign of the dissipation to the stability of the scheme $dG[u_0]$.

PROPOSITION 2.1. *If the scheme G is consistent with Eqs.* (1.1) *and satisfies Assumptions* 2.1 *and* 2.3 *with negative dissipation as in* (2.14), *then the linearized evolutionary problem $dG[u_0]u(x) = 0$ is unstable* (*i.e., violates the von Neumann condition*).

Proof. We consider the characteristic equation $\det d\hat{G}[u_0](\xi) = 0$. It is enough to show that the above equation has a solution ξ with real $\xi_- = (\xi_1, \ldots, \xi_{m-1})$ and complex ξ_m such that $\operatorname{Re} i\xi_m > 0$. Without loss we may assume that the matrix $df[u_0]$ is in the block form $\operatorname{diag}(0, A)$, where $\det A \neq 0$, and $l_0(u_0) = (r_0(u_0))^T = (1, 0, \ldots, 0)$. Let $\xi_- = \xi_0 \cdot s_-$, where $s_- = (s_1, s_2, \ldots, s_{m-1})$, and represent $\xi_m = s_m \xi_0 + \xi_m' \cdot \xi_0^{k+1}$. By consistency (and $\partial_u f_m(u) = I$)

$$d\hat{G}[u_0](\xi) = i\xi_m' \cdot \xi_0^{k+1} \cdot I + d\hat{G}[u_0](s\xi_0) + O(\xi_0^{k+2})$$

and by (2.4) and (2.5) the first column of $d\hat{G}[u_0](s\xi_0)$ is $O(\xi_0^{k+1})$. Divide the first column of $d\hat{G}[u_0](\xi)$ by ξ_0^{k+1} and denote the resulting matrix function by $B(\xi_0, \xi_m')$. Then, by (2.12)

$$\det B(\xi_0, \xi_m') = (i\xi_m' + b(u_0) i^{k+1}) \det A + O(\xi_0).$$

Therefore

$$\xi_m = s_m \xi_0 - b(u_0) i^k \xi_0^{k+1} + O(\xi_0^{k+2}) \qquad \text{and} \qquad \xi_- = \xi_0 \cdot s_- \tag{2.15}$$

satisfy the characteristic equation $\det d\hat{G}[u_0](\xi) = 0$. Obviously for small ξ_0, $\operatorname{Re} i\xi_m > 0$ if (2.14) holds. Q.E.D.

For stationary problems (1.1) we have a similar result.

PROPOSITION 2.2. *Let $\tilde{G}(\{E_x^j E_t^{j_{m+1}} u(x,t)\}) = 0$ be a difference scheme consistent with the system $u_t + \sum_{i=1}^m (f_i(u))_{x_i} = 0$. Suppose that the operator $\tilde{G}$ when restricted to time independent functions $u(x)$ coincides with the operator G in (1.9). If the scheme G satisfies the conditions of Proposition 2.1, the scheme $d\tilde{G}[u_0]$ is unstable.*

The proof of the proposition is similar to the previous one. Let $d\hat{\tilde{G}}[u_0](\xi, \xi_{m+1})$ be the symbol of $d\tilde{G}[u_0]$. Then, by consistency,

$$d\hat{\tilde{G}}[u_0](s\xi_0, \xi_{m+1}) = i\xi_{m+1} \cdot I + d\hat{G}[u](s\xi_0) + O(\xi_{m+1}^2) + O(\xi_0 \cdot \xi_{m+1})$$

so that instead of (2.15) we conclude that

$$\xi_{m+1} = -b(u_0) i^k \xi_0^{k+1} + O(\xi_0^{k+2}), \qquad \xi = \xi_0 \cdot s \tag{2.16}$$

is a solution of the characteristic equation for $d\tilde{G}[u_0]$.

The following two propositions provide clear sufficient conditions which assure that Assumptions 2.1–2.3 with (2.13) are satisfied.

PROPOSITION 2.3. *If the problem in (1.1) is evolutionary, the scheme $dG[u_0]$ for $u_0 \in S$ is kth order accurate (as in (2.6)) and $(k+1)$th order dissipative (i.e., the solutions ξ_m of the equation $\det d\hat{G}[u_0](\xi_-, \xi_m) = 0$ have $\operatorname{Im} \xi_m \geq \delta |\xi_-|^{k+1}$ for $|\xi_i| \leq \pi$ $(i = 1, \ldots, m-1)$ then Assumptions 2.1–2.3 are satisfied with positive dissipation as in (2.13).*

PROPOSITION 2.4. *Let the scheme $\tilde{G}$ be defined as in Proposition* 2.2 *and suppose that the linearized scheme $d\tilde{G}[u_0]$ for $u_0 \in S$ is kth order accurate and $(k+1)$th order dissipative. Then Assumptions* 2.1–2.3 *with* (2.13) *are satisfied.*

Proof. Obviously, in both propositions the symbol $d\hat{G}[u](\xi)$ satisfies the conditions (2.6) and (2.8) so that Assumptions 2.1 and 2.2 follow. Assumption 2.3 with positive dissipativity then follows from (2.15) or (2.16) and the dissipativity of the schemes $dG[u_0]$ and $d\tilde{G}[u_0]$.

Consider now the linearization of the operator $G_0(\{E_\tau^{j_0}u(\tau)\})$

$$dG_0[u_0] = \sum C_{j_0}(u_0)E_\tau^{j_0}, \qquad j_0 \in J_0, \quad u_0 \in S \tag{2.17}$$

where

$$C_{j_0}(u_0) = \left.\frac{\partial G_0(\{u_{j_0}\})}{\partial u_{j_0}}\right|\{u_{j_0}\} = \{u_0\}. \tag{2.18}$$

By the definition in (1.17), $dG_\tau[u_0] = (E_\tau - I)dG_0[u_0]$ so that

$$d\hat{G}_0[u_0](\xi_0) = d\hat{G}_\tau[u_0](\xi_0)/(e^{i\xi_0} - 1). \tag{2.19}$$

Assumptions 2.1–2.3 imply immediately

PROPOSITION 2.5. *The symbol $d\hat{G}_0[u_0](\xi_0)$ for $u_0 \in S$ satisfies the conditions*

$$\det d\hat{G}_0[u_0](\xi_0) \neq 0 \qquad \text{for } \xi_0 \neq 0 \bmod 2\pi, \tag{2.20}$$

$$d\hat{G}_0[u_0](\xi_0) \cdot r_0(u_0) = O(\xi_0^k) \tag{2.21}$$

and

$$l_0(u_0)d\hat{G}_0[u_0](\xi_0) \cdot r_0(u_0) = b(u_0)(iq\xi_0)^k + O(\xi_0^{k+1}). \tag{2.22}$$

For small positive μ_0 consider the map

$$(u_0, \mu) \to u_L(u_0, \mu) = u_0 + \mu r_0(u_0), \qquad \text{where } u_0 \in S,\ -\mu_0 < \mu < \mu_0. \tag{2.23}$$

The above map is one-to-one and the image of it is a tubular neighborhood of S. Let

$$\varepsilon = q^{-1}\left|\lambda_0(u_L)/b(u_0)\right|^{1/k}, \qquad \text{where } u_L = u_L(u_0, \mu). \tag{2.24}$$

We state now the

MAIN THEOREM. *Let the scheme G in* (1.9) *be conservative and consistent with* (1.1) *and satisfy Assumptions* 2.1–2.3 *with $k = 1$ or $k = 3$. Then for any u_L which is close to the surface S and satisfies $\lambda_0(u_L) > 0$ for positive* *dissipation or* $\lambda_0(u_L) < 0$ *for the negative one, there exists a smooth trajectory* $u(\tau)$, $-\infty < \tau < \infty$, *depending smoothly on* u_L *with* $\lim_{\tau \to -\infty} u(\tau) = u_L$, $\lim_{\tau \to +\infty} u(\tau) = u_R = \varphi(u_L)$ *such that*

(i) *For any* τ_0 *the grid function*

$$u_j = u(\tau_0 + \varepsilon j), \qquad j \in \mathbf{Z}, \tag{2.25}$$

with ε *as in* (2.24) *is a discrete shock solution of* (1.9).

(ii) *The trajectory* $u(\tau)$ *is given by the formula*

$$u(\tau) = \frac{u_L + u_R}{2} + \left(\lambda_0(u(\tau))/\lambda_0(u_L)\right) \cdot \frac{u_L - u_R}{2} + O\left(|u_L - u_R|^2\right) \tag{2.26}$$

and the function $i^{k-1} \cdot \lambda_0(u(\tau))/\lambda_0(u_L)$ *tends uniformly to the odd solution* $y(\tau)$ *of the problem*

$$\frac{d^k y}{d\tau^k} = \frac{1}{2}(y^2 - 1), \qquad \lim_{\tau \to -\infty} y(\tau) = i^{k-1}, \tag{2.27}$$

as u_L *tends to the surface* S.

(iii) *For* $k = 1$ *the discrete shock solution is locally unique*; *i.e., any such solution which is located in a small neighbourhood of a point* $u_0 \in S$ *should coincide with one of the grid functions in* (2.25). *The function* $\lambda_0(u(\tau))$ *is monotonic so that the solution in* (2.25) *is uniquely determined by a value of* λ_0 *at* u_j.

(iv) *For* $k = 3$ *the discrete shock solution is possibly locally non-unique*[3]; *however, the solution is unique "micro-locally"* (*see Remark* 4.1). *The grid function in* (2.25) *is uniquely determined by the values of* λ_0 *at any three consequent points* u_j, u_{j+1}, u_{j+2}.

Remark 2.1. Recall that the grid function u_j in (2.25) is defined on a grid with the mesh size h/q so that the original size h corresponds to the step $|\lambda_0(u_L)/b(u_0)|^{1/k}$ along the trajectory $u(\tau)$, which is independent of the denominator q. The solution $y(\tau)$ of (2.27) while oscillating (when $k = 3$) tends exponentially to the constant states ± 1 as $\tau \to \infty$. We call the domain $|\tau| < R$ a shock layer if outside of this domain $y(\tau)$ differs from the corresponding infinite state less than by δ, where δ is some prescribed small number (say $\delta = 0.01$). As one would expect, the corresponding shock layer in the physical plane x has a width proportional to $h/|\lambda_0(u_L)/b(u_0)|^{1/k}$;

[3] *Note added in proof*: Recently C. K. McCord (preprint, Dept. of Math., Univ. of Wisconsin, Madison, 1983) has proved that Eq. (2.27) for $k = 3$ has a *unique* bounded solution.

i.e., the width decreases when the shock is *stronger* ($\lambda_0(u_L)$ is greater) or the dissipation is *weaker* ($b(u_0)$ smaller).

3. Reduction to the Central Manifold

In this section we reduce Eq. (1.19) to the central manifold and then rescale the resulting equations. By shifting the function $u(\tau)$ we may assume that the index j_0 in (1.19) varies over $0 \le j_0 \le N$. For convenience we also replace j_0 by j and rewrite (1.19) as

$$G_0\big(u(\tau), \dots, E_\tau^j u(\tau), \dots, E_\tau^N u(\tau)\big) - f(u_L) = 0. \tag{3.1}$$

We consider u_L in (3.1) as a function $u_L(u_0, \mu) = u_0 + \mu r_0(u_0)$. Define the grid functions

$$w(\tau) = \begin{pmatrix} w^{(0)}(\tau) \\ \vdots \\ w^{(N-1)}(\tau) \end{pmatrix}, \qquad \text{where } w^{(j)}(\tau) = E_\tau^j u(\tau) - u_0 \tag{3.2}$$

and

$$\tilde{w}(\tau) = \begin{pmatrix} w(\tau) \\ \mu \end{pmatrix}. \tag{3.3}$$

It is convenient to consider μ as a constant grid function, while $u_0 \in S$ is a parameter. The grid function $\tilde{w}(\tau)$ satisfies the equations

$$\begin{aligned} &\text{(i)} && G_0\big(w^{(0)}(\tau) + u_0, E_\tau w(\tau) + \bar{u}_0\big) - f\big(u_0 + \mu r(u_0)\big) = 0 \\ &\text{(ii)} && E_\tau w^{(j)}(\tau) - w^{(j+1)}(\tau) = 0, \qquad j = 0, 1, \dots, N-2 \\ &\text{(iii)} && E_\tau \mu(\tau) - \mu(\tau) = 0. \end{aligned} \tag{3.4}$$

Here, by $\bar{u}_0$ or in general by $\bar{u}$ we denote the vector $\left.\begin{pmatrix} u \\ \vdots \\ u \end{pmatrix}\right\} N$ times. We shall denote the left-hand side of (3.4) by $H(\tilde{w}(\tau), E_\tau \tilde{w}(\tau); u_0)$ and study the function $H(\tilde{w}_1, \tilde{w}_2; u_0)$ of two vector variables $\tilde{w}_1, \tilde{w}_2 \in R^{nN+1}$ and parameter $u_0 \in S$. We say that $\tilde{w}$ is a fixed point of the function H if $H(\tilde{w}, \tilde{w}; u_0) = 0$. It follows from (3.4ii) that $\tilde{w} = \binom{\bar{u}}{\mu}$, and then by (1.18), $f(u + u_0) = f(u_0 + \mu r_0(u_0))$, so that

$$\tilde{w}_L = \begin{pmatrix} \mu \bar{r}_0(u_0) \\ \mu \end{pmatrix} \qquad \text{and} \qquad \tilde{w}_R = \begin{pmatrix} \bar{u}_R - \bar{u}_0 \\ \mu \end{pmatrix}, \tag{3.5}$$

where

$$u_R = \varphi(u_L) = \varphi\big(u_0 + \mu r_0(u_0)\big)$$

are *locally* the only fixed points of the function H for $u_0 \in S$ and small μ.

Define the matrix functions

$$\tilde{A}(u_0) = \begin{pmatrix} A(u_0) & 0 \\ 0 & -1 \end{pmatrix} = \frac{\partial H(\tilde{w}_1, \tilde{w}_2; u_0)}{\partial \tilde{w}_1}\bigg|_{\tilde{w}_1 = \tilde{w}_2 = 0} \quad (3.6)$$

and

$$\tilde{B}(u_0) = \begin{pmatrix} B(u_0) & 0 \\ 0 & 1 \end{pmatrix} = \frac{\partial H(\tilde{w}_1, \tilde{w}_2; u_0)}{\partial \tilde{w}_2}\bigg|_{\tilde{w}_1 = \tilde{w}_2 = 0}, \quad (3.7)$$

where

$$A(u_0) = \begin{pmatrix} C_0(u_0) & & & \bigcirc \\ & -I & & \\ & & \ddots & \\ \bigcirc & & & -I \end{pmatrix},$$

$$B(u_0) = \begin{pmatrix} C_1(u_0) & C_2(u_0) & \cdots & C_N(u_0) \\ I & 0 & & \\ & \ddots & & \\ \bigcirc & & I & 0 \end{pmatrix}, \qquad u_0 \in S \quad (3.8)$$

and $C_j(u_0)$, $j = 0, 1, \ldots, N$, is defined as in (2.18). Consider the λ-matrix or the pencil of matrices $A(u_0) + \lambda B(u_0)$. The characteristic equation for that pencil is

$$\det(A(u_0) + \lambda B(u_0)) = \det\left(\sum_{j=0}^{N} C_j(u_0)\lambda^j \right) = 0. \quad (3.9)$$

Obviously, for $\lambda = e^{i\xi_0}$

$$\sum_{j=0}^{N} C_j(u_0)\lambda^j = d\hat{G}_0[u_0](\xi_0) \quad (3.10)$$

so that in view of (2.20) Eq. (3.9) has no solutions λ on the unit circle $|\lambda| = 1$, besides $\lambda = 1$. From (2.21) and (2.22) we also derive

$$\sum_{j=0}^{N} C_j(u_0)\lambda^j r_0(u_0) = O\big((\lambda - 1)^k\big) \quad (3.11)$$

and

$$l_0(u_0) \sum_{j=0}^{N} C_j(u_0)\lambda^j r_0(u_0) = b(u_0) q^k (\lambda - 1)^k + O\big((\lambda - 1)^{k+1}\big),$$
$$b(u_0) \neq 0. \quad (3.12)$$

Since, by (1.18),

$$\sum_{j=0}^{N} C_j(u_0) = df[u_0], \tag{3.13}$$

one can easily deduce from (3.11) and (3.12) that $\lambda = 1$ is an eigenvalue of $A(u_0) + \lambda B(u_0)$ of multiplicity k. The corresponding eigenspace, i.e., $\operatorname{Ker}(A(u_0) + B(u_0))$, is spanned by the vector $\bar{r}_0(u_0)$. (Here again $\bar{r}_0(u_0)$ is N times repeated vector $r_0(u_0)$.) The vector function

$$\rho(\lambda) = \begin{pmatrix} r_0(u_0) \\ \lambda r_0(u_0) \\ \vdots \\ \lambda^{N-1} r_0(u_0) \end{pmatrix} \tag{3.14}$$

satisfies the relation

$$(A(u_0) + \lambda B(u_0))\rho(\lambda) = \begin{pmatrix} \sum C_j(u_0)\lambda^j r_0(u) \\ 0 \end{pmatrix} = O(\lambda - 1)^k. \tag{3.15}$$

Thus, $\rho(\lambda)$ is a root function of $A(u) + \lambda B(u)$ of multiplicity k corresponding to the eigenvalue $\lambda = 1$, and the vectors

$$\rho_i = \frac{1}{i!}\frac{d^i\rho}{d\lambda^i}(1), \qquad i = 0, 1, \ldots, k-1 \tag{3.16}$$

form a Jordan chain corresponding to the eigenvalue $\lambda = 1$. (For the definition of root function and Jordan chain see [1].) This implies that the matrix

$$X_0(u_0) = (\rho_0, \rho_1, \ldots, \rho_{k-1}) \tag{3.17}$$

satisfies the identity

$$A(u_0)X_0(u_0) + B(u_0)X_0(u_0)J(1) = 0, \tag{3.18}$$

where $J(1)$ is a simple Jordan cell with the eigenvalue $\lambda = 1$. Now consider the real mutually orthogonal projectors

$$\begin{aligned}
P^{c}(u_0) &= (2\pi i)^{-1}\oint_{|\lambda - 1| = \delta} (A(u_0) + \lambda B(u_0))^{-1} B(u_0)\, d\lambda \\
P^{s}(u_0) &= (2\pi i)^{-1}\oint_{|\lambda| = 1 - \delta} (A(u_0) + \lambda B(u_0))^{-1} B(u_0)\, d\lambda \\
P^{u}(u_0) &= (2\pi i)^{-1}\oint_{|\lambda| = 1 - \delta} (\lambda A(u_0) + B(u_0))^{-1} A(u_0)\, d\lambda,
\end{aligned} \tag{3.19}$$

where δ is sufficiently small so that no eigenvalues of $A(u_0) + \lambda B(u_0)$ besides $\lambda = 1$ are located in the ring $1 - \delta \leq |\lambda| \leq (1 - \delta)^{-1}$. (The letters c, s, and u come from central, stable, and unstable.) Denote by $W^{\mathrm{c}}(u_0), W^{\mathrm{s}}(u_0), W^{\mathrm{u}}(u_0)$ the images of the corresponding projectors. Then by $\tilde{W}^{\mathrm{s}}(u_0)$ and $\tilde{W}^{\mathrm{u}}(u_0)$ we denote the space of $(Nn + 1)$-dimensional vectors $\tilde{w} = \binom{w}{0}$, $w \in W^{\mathrm{s}}(u_0)$ or $w \in W^{\mathrm{u}}(u_0)$ correspondingly, while $\tilde{W}^{\mathrm{c}}(u_0)$ consists of all vectors $\tilde{w} = \binom{w}{\mu}$ with $w \in W^{\mathrm{c}}(u_0)$ and $\mu \in R^1$. Obviously, one can select a basis in the above spaces which depends locally smoothly on u_0. However, we consider a rather general situation in which S is not necessarily homeomorphic to a ball. It may happen that there is no global smooth basis in $\tilde{W}^{\mathrm{s}}(u_0)$ and $\tilde{W}^{\mathrm{u}}(u_0)$ for all $u_0 \in S$. Therefore, we shall develop an invariant approach which is independent of a particular basis in $\tilde{W}^{\mathrm{s}}(u_0)$ and $\tilde{W}^{\mathrm{u}}(u_0)$. We start with some definitions pertaining to pencils of linear transformations $A + \lambda B$.

Let V and $\tilde{V}$ be n-dimensional real vector spaces and A and B linear transformations from V to $\tilde{V}$. Assume that the pencil $A + \lambda B$ is regular; i.e., the transformation $A + \lambda B$ is one to one for some value of λ. The spectrum of the pencil $A + \lambda B$ in the complex sphere $\mathbf{C} \cup \infty$ consists of the values of $\lambda \in \mathbf{C}$ for which $A + \lambda B$ (acting on complexification of V and $\tilde{V}$) in singular, and of $\lambda = \infty$ if B is singular. When A and B are represented by matrices, the finite spectrum is given by the equation $\det(A + \lambda B) = 0$. The subspace $U \subset V$ is called an invariant subspace of the pencil if AU and BU belong to the same subspace $\tilde{U} \subset \tilde{V}$ and $\dim U = \dim \tilde{U}$. Denote the restrictions of A and B to U by $A|U$ and $B|U$ respectively. Then the restricted pencil $A|U + \lambda B|U$ is also regular.

Definition 3.1. The invariant subspace U of the pencil $A + \lambda B$ is called stable, central-stable, central, central-unstable, and unstable, if the spectrum of the restricted pencil $A|U + \lambda B|U$ belongs correspondingly to the sets $|\lambda| < 1$, $|\lambda| \leq 1$, $|\lambda| = 1$, $|\lambda| \geq 1$, and $|\lambda| > 1$.

For central-stable subspaces U there exists a unique linear transformation $M: U \to U$ such that

$$A|U + (B|U)M = 0. \tag{3.20}$$

The spectrum of M coincides with the spectrum of the pencil $A|U + \lambda B|U$. Similarly, for the central-unstable subspace U the transformation $M: U \to U$ satisfies the identity

$$(A|U)M + B|U = 0 \tag{3.21}$$

and the spectrum of M coincides with the spectrum of the pencil $A|U + \lambda B|U$, which is the reciprocal of the spectrum of the pencil $A|U + \lambda B|U$.

In view of the above definitions it is easy to show that $W^{c}(u_0)$, $W^{s}(u_0)$, and $W^{u}(u_0)$ are correspondingly central, stable, and unstable invariant subspaces of the pencil $A(u_0) + \lambda B(u_0)$, while $\tilde{W}^{c}(u_0)$, $\tilde{W}^{s}(u_0)$ and $\tilde{W}^{u}(u_0)$ are the corresponding invariant subspaces of $\tilde{A}(u_0) + \lambda \tilde{B}(u_0)$. In view of (3.18) it is obvious that the vectors ρ_i, $i = 0, 1, \dots, k-1$, in (3.16) form a basis of $W^{c}(u_0)$, while $J(1)$ is the corresponding transformation M of (3.20) written in this basis. The spaces $\tilde{W}^{c}(u_0)$, $\tilde{W}^{s}(u_0)$ and $\tilde{W}^{u}(u_0)$ with u_0 varying over S form smooth vector bundles over S which we denote by $\tilde{W}^{c}$, $\tilde{W}^{s}$, and $\tilde{W}^{u}$. These bundles are smooth subbundles of the trivial bundle $\tilde{W} = R^{nN+1} \times S$, which is the direct sum

$$\tilde{W} = \tilde{W}^{s} \oplus \tilde{W}^{c} \oplus \tilde{W}^{u}. \tag{3.22}$$

The function $H(\tilde{w}_1, \tilde{w}_2; u_0)$ could be considered as a map of the bundle $\tilde{W} \oplus \tilde{W}$ into $\tilde{W}$, which transfers the point $(\tilde{w}_1, \tilde{w}_2)$ of the fiber $R^{nN+1} \oplus R^{nN+1}$ over u_0 into the point $H(\tilde{w}_1, \tilde{w}_2; u_0)$ of the fiber R^{nN+1} over u_0. This map, which is denoted again by H, satisfies the conditions of Theorem 5.2. The correspondence with the notations of the theorem is as follows: $\tilde{W} \leftrightarrow W$, $\tilde{A}(u_0) + \lambda \tilde{B}(u_0) \leftrightarrow A(p) + \lambda B(p)$, $H \leftrightarrow H$, $\tilde{g} \leftrightarrow g$, $\tilde{F} \leftrightarrow F$, and $\tilde{\pi} \leftrightarrow \pi$. Thus, there exists a central manifold, which is the image of the map $\tilde{g}\colon \tilde{W}^{c} \to \tilde{W}$, and a map $\tilde{F}\colon \tilde{W}^{c} \to \tilde{W}^{c}$ such that the compositions of $\tilde{g}$ with the natural projectors $\tilde{\pi}^{s}$, $\tilde{\pi}^{u}$, and $\tilde{\pi}^{c}$ satisfy

$$\begin{gathered} \tilde{\pi}^{c} \circ g = \text{identity}, \\ \tilde{\pi}^{s} \circ \tilde{g}(0) = \tilde{\pi}^{u} \circ \tilde{g}(0) = \tilde{\pi}^{s} \circ \partial \tilde{g}(0) = \tilde{\pi}^{u} \circ \partial \tilde{g}(0) = 0 \end{gathered} \tag{3.23}$$

and equation $H(\tilde{w}_1, \tilde{w}_2; u_0) = 0$ on the central manifold is equivalent to

$$\tilde{w}_2 = \tilde{g} \circ \tilde{F} \circ \tilde{\pi}^{c}(\tilde{w}_1). \tag{3.24}$$

In particular, the identity

$$H(\tilde{g}, \tilde{g} \circ \tilde{F}) = 0 \tag{3.25}$$

holds.

The space $W^{c}(u_0)$ has a basis ρ_i, $i = 1, \dots, k$, which is defined for all $u_0 \in S$ and depends smoothly on u_0. We denote the coordinates of a point $w \in W^{c}(u_0)$ in this basis by $y = (y^{(1)}, \dots, y^{(k)})^{T}$, while the points $\tilde{w} \in \tilde{W}^{c}(u_0)$ have coordinates (y, μ). The map $\tilde{F}\colon \tilde{W}^{c} \to \tilde{W}^{c}$ in these coordinates looks like

$$(y_2, \mu_2) = \tilde{F}(y_1, \mu_1; u_0). \tag{3.26}$$

According to (3.4iii), μ_1 and μ_2 satisfy the equation $\mu_1 = \mu_2$. From now on we shall consider μ again as a parameter and rewrite Eq. (3.26) as

$$y_2 = F(y_1, \mu, u_0) \tag{3.27}$$

while the map $\tilde{g}: \tilde{W}^c \to \tilde{W}$ is represented by a function

$$w = g(y, \mu, u_0) = X_0(u_0)y + O(|y|^2 + \mu^2). \tag{3.28}$$

The function $F(y, \mu, u_0)$ vanishes at $y = 0$, $\mu = 0$ and has the following Taylor expansion in the variables y and μ:

$$F(y, \mu, u_0) = J(1)y + O(|y|^2 + \mu^2). \tag{3.29}$$

We need to know only quadratic terms for the last component $F^{(k)}$ which include the products $(y^{(1)})^2$, μ^2, and $y^{(1)}\mu$. For small μ the function H has locally only two fixed points, which are $\tilde{w}_L$ and $\tilde{w}_R$ in (3.5). By part (iii) of Theorem 5.2 these fixed points should belong to the central manifold of H. The y coordinates of the points $\tilde{w}_L$ and $\tilde{w}_R$ are correspondingly

$$y_L(\mu, u_0) = \mu e, \qquad y_R(\mu, u_0) = -\mu e + O(\mu^2),$$
$$\text{where } e = (1, 0, \ldots, 0)^T \in R^k. \tag{3.30}$$

(The formula for $y_R(\mu, u_0)$ follows trivially from (3.5) and (1.6).) Since the fixed points of F are also fixed points of H (and vice versa), $y_L(\mu, u_0)$ and $y_R(\mu, u_0)$ are locally the *only* fixed points of F for small μ. For the kth component of the vector function F we write the expansion

$$F^{(k)}\left(y^{(1)}e, \mu, u_0\right) = \tfrac{1}{2}a_1(u_0)\left(y^{(1)}\right)^2 + a_2(u_0)\mu y^{(1)}$$
$$+ \tfrac{1}{2}a_3(u_0)\mu^2 + \text{higher order terms}. \tag{3.31}$$

Substituting the fixed points $y_L(\mu, u_0)$ and $y_R(\mu, u_0)$ one derives easily that

$$a_2(u_0) = 0 \qquad \text{and} \qquad a_3(u_0) = -a_1(u_0). \tag{3.32}$$

PROPOSITION 3.1. *The coefficient* $a_1(u_0)$ *in* (3.31) *is equal to* $-(q^k b(u_0))^{-1}$, *where* $b(u_0)$ *is the coefficient of dissipativity in* (2.9).

Proof. Fix $\mu = 0$ and u_0 and set $y_1 = \varepsilon e$, $y_2 = F(y_1, 0, u_0) = y_1 + \varepsilon^2\alpha + O(\varepsilon^3)$. Here $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})^T \in R^k$ and $\alpha^{(k)} = \frac{1}{2}a_1(u_0)$. In view of (3.25) the points

$$\tilde{w}_1 = \begin{pmatrix} w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} g(y_1, 0, u_0) \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{w}_2 = \begin{pmatrix} w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} g(y_2, 0, u_0) \\ 0 \end{pmatrix} \tag{3.33}$$

satisfy the equation $H(\tilde{w}_1, \tilde{w}_2; u_0) = 0$. By (3.28)

$$g(y, 0, u_0) = X_0(u_0)y + O(|y|^2) \tag{3.34}$$

so that

$$w_2 = g\big(y_1 + \varepsilon^2\alpha, 0, u_0\big) + O(\varepsilon^3) = w_1 + \varepsilon^2\beta + O(\varepsilon^3), \qquad \beta = X_0(u_0)\alpha. \tag{3.35}$$

Using the definition of $X_0(u_0)$ in (3.17) and (3.16) we can represent the vector β as

$$\beta = p(d/d\lambda)\cdot\rho(\lambda)|_{\lambda=1}, \qquad \text{where } p(d/d\lambda) = \sum_{i=0}^{k-1}\frac{\alpha^{(i+1)}}{i!}\left(\frac{d}{d\lambda}\right)^i.$$

Equations (3.4ii) imply

$$w_2^{(j)} - w_1^{(j+1)} = 0, \qquad \text{or} \qquad w_1^{(j+1)} = w_1^{(j)} + \varepsilon^2\beta^{(j)} + O(\varepsilon^3),$$
$$j = 0, 1, \dots, N-2.$$

Note that the components $\beta^{(j)} = r_0(u_0)p(d/d\lambda)\lambda^j|_{\lambda=1}$, $j = 0, 1, \dots, N-1$, so that

$$w_1^{(j+1)} = w_1^{(0)} + \varepsilon^2 r_0(u_0)\left[p(d/d\lambda)\frac{\lambda^{j+1}-1}{\lambda-1}\right]_{\lambda=1} + O(\varepsilon^3),$$
$$j = 0, 1, \dots, N-2,$$

and (3.35) implies

$$w_2^{(j)} = w_1^{(0)} + \varepsilon^2 r_0(u_0)\left[p(d/d\lambda)\frac{\lambda^{j+1}-1}{\lambda-1}\right]_{\lambda=1} + O(\varepsilon^3),$$
$$j = 0, 1, \dots, N-1. \tag{3.36}$$

Equation (3.4i) in variables w_1 and w_2 reads

$$G_0\big(u_0 + w_1^{(0)}, \bar{u}_0 + w_2\big) - f(u_0) = 0. \tag{3.37}$$

Using the identity (1.18) and substituting (3.36) in (3.37) we get

$$f\big(u_0 + w_1^{(0)}\big) + \varepsilon^2\left[\sum_{j=0}^{N} C_j(u_0)r_0(u_0)p(d/d\lambda)\frac{\lambda^j-1}{\lambda-1}\right]_{\lambda=1}$$
$$-f(u_0) = O(\varepsilon^3). \tag{3.38}$$

Multiply the last equation from the left by the vector $l_0(u_0)$. By formula

(3.12),

$$l_0(u_0)\sum_{j=0}^{N} C_j(u_0)\frac{\lambda^j-1}{\lambda-1}r_0(u_0) = l_0(u_0)\sum_{j=0}^{N} C_j(u_0)\frac{\lambda^j}{\lambda-1}r_0(u_0)$$
$$= b(u_0)q^k(\lambda-1)^{k-1} + O\big((\lambda-1)^k\big), \tag{3.39}$$

where the first equality follows from (3.13). Since $w_1^{(0)}$ by (3.33) and (3.34) is equal to $\varepsilon r_0(u_0) + O(\varepsilon^2)$, we have by (2.11)

$$\begin{aligned} & l_0(u_0)\big(f(u_0 + w_1^{(0)}) - f(u_0)\big) \\ &= l_0(u_0)\left(df[u_0]w_1^{(0)} + \frac{\varepsilon^2}{2}d^2f[u_0](r_0(u_0))\right) + O(\varepsilon^3) \\ &= \frac{\varepsilon^2}{2} + O(\varepsilon^3). \end{aligned} \tag{3.40}$$

Combining (3.39) and (3.40) we get finally

$$\begin{aligned} & \tfrac{1}{2} + \Big[p(d/d\lambda)\big(b(u_0)q^k(\lambda-1)^{k-1} + O\big((\lambda-1)^k\big)\big)\Big]_{\lambda=1} \\ &= \tfrac{1}{2} + \alpha^{(k)}b(u_0)q^k = 0. \end{aligned} \tag{3.41}$$

Q.E.D.

Now we can rescale Eq. (3.27) in the same way as in [3]. Namely, let

$$\varepsilon = |\mu a_1(u_0)|^{1/k} = q^{-1}|\mu/b(u_0)|^{1/k}, \tag{3.42}$$

which differs from ε in (2.24) by $O(\mu^2)$, and define a new vector variable $y = (y^{(1)}, \ldots, y^{(k)})^T$ by

$$y_{\text{new}}^{(i)} = y_{\text{old}}^{(i)} \cdot a_1(u_0)/\varepsilon^{k+i-1}, \qquad i = 1, 2, \ldots, k. \tag{3.43}$$

Substituting (3.43) in Eq. (3.27), we get in the new variables

$$y_2 = y_1 + \varepsilon\Big[J(0)y_1 + \tfrac{1}{2}\big(0, 0, \ldots, 0, \big(y_1^{(1)}\big)^2 - 1\big)^T + O(\varepsilon)\Big] = F(y_1, \varepsilon, u_0), \tag{3.44}$$

where $J(0)$ is the nilpotent Jordan cell. We denote the right-hand side of (3.44) again by $F(y_1, \varepsilon, u_0)$ and the expression in the square brackets by

$\hat{F}(y_1, \varepsilon, u_0)$. The fixed points $y_L(\mu, u_0)$ and $y_R(\mu, u_0)$ in (3.30) now become

$$y_L(\varepsilon, u_0) = \mathrm{sign}(-\mu b(u_0))e, \qquad y_R(\varepsilon, u_0) = \mathrm{sign}(\mu b(u_0))e + O(\mu). \tag{3.45}$$

These are the only fixed points of $F(y, \varepsilon, u_0)$ for small ε. We choose the sign of μ so that

$$\mathrm{sign}(\mu b(u_0)) = (-1)^{(k+1)/2}. \tag{3.46}$$

The new y coordinates of the grid function $w(\tau)$ satisfy the equation

$$\frac{E_\tau y(\tau) - y(\tau)}{\varepsilon} = \hat{F}(y(\tau), \varepsilon, u_0) \tag{3.47}$$

with the condition on infinity

$$\begin{aligned} \lim_{\tau \to -\infty} y(\tau) &= y_L(\varepsilon, u_0) = (-1)^{(k-1)/2} e, \\ \lim_{\tau \to +\infty} y(\tau) &= y_R(\varepsilon, u_0) = (-1)^{(k+1)/2} e + O(\mu). \end{aligned} \tag{3.48}$$

If one makes the change of variable

$$\tau_{\mathrm{new}} = \tau_{\mathrm{old}} \cdot \varepsilon/(h/q), \tag{3.49}$$

the shift operator $E_\tau u(\tau) = u(\tau + h/q)$ becomes

$$E_\tau u(\tau) = u(\tau + \varepsilon) \tag{3.50}$$

so that one can interpret the problem in (3.47) and (3.48) as a one-step difference approximation of the canonical problem

$$\frac{dy(\tau)}{d\tau} = \hat{F}(y(\tau), 0, u_0) = J(0)y + \tfrac{1}{2}\left(0, 0, \ldots, (y^{(1)})^2 - 1\right)^T \tag{3.51}$$

with conditions on infinity

$$y(-\infty) = (-1)^{(k-1)/2} \cdot e, \qquad y(+\infty) = (-1)^{(k+1)/2} \cdot e. \tag{3.52}$$

4. Proof of the Main Theorem

First we consider the easy case $k = 1$. According to (3.46), $\mu > 0$ in the case of positive dissipation as in (2.13), and $\mu < 0$ if the dissipation is negative. Then (3.43) becomes

$$y_{\text{new}} = y_{\text{old}}/\mu, \tag{4.1}$$

and the fixed points in the new coordinates are

$$y_L(\varepsilon, u_0) = 1, \qquad y_R(\varepsilon, u_0) = -1 + O(\varepsilon). \tag{4.2}$$

Equation (3.47) in the new τ variable becomes

$$y(\tau+\varepsilon) = y(\tau) + \varepsilon\hat{F}(y(\tau), \varepsilon, u_0) = y(\tau) + \varepsilon\big[(y^2(\tau)-1)/2 + O(\varepsilon)\big]. \tag{4.3}$$

The function $\hat{F}(y, \varepsilon, u_0)$ vanishes only at y_L and y_R as above and is negative for $y \in (y_R, y_L)$. Moreover, for $y_R < y(\tau) < y_L$, we get

$$y_R < y(\tau+\varepsilon) < y(\tau) < y_L. \tag{4.4}$$

Indeed,

$$y(\tau) - y(\tau+\varepsilon) = -\varepsilon\big(\hat{F}(y(\tau), \varepsilon, u_0) - \hat{F}(y_R, \varepsilon, u_0)\big) < y(\tau) - y_R$$

for small ε. Equation (4.3) could be resolved for $y(\tau)$,

$$y(\tau-\varepsilon) = y(\tau) - \varepsilon\big[(y^2(\tau)-1)/2 + O(\varepsilon)\big], \tag{4.5}$$

where the function in the square brackets vanishes only at y_L and y_R. As previously, one can show that $y(\tau) \in (y_R, y_L)$ implies

$$y_R < y(\tau) < y(\tau-\varepsilon) < y_L.$$

Thus, for any $y(\tau_0) \in (y_R, y_L)$ the sequence

$$y_j = y(\tau_0 + j\varepsilon), \qquad j \in \mathbf{Z} \tag{4.6}$$

defined by (4.3) for $j > 0$ and by (4.5) for $j < 0$ is decreasing as j increases and $\lim_{j\to\infty} y_j = y_R$, $\lim_{j\to-\infty} y_j = y_L$. We parametrize the interval (y_R, y_L) by $\tau \in R^1$ as follows. Set $y(0) = 0$ and parametrize the interval $[0, y(\varepsilon)]$ by a smooth decreasing function $y_\varepsilon(\tau)$, $\tau \in [0, \varepsilon]$. Then define $y_\varepsilon(\tau + j\varepsilon)$, $\tau \in [0, \varepsilon]$, $j \in \mathbf{Z}$ by (4.3) and (4.5). For a suitable parametrization of the interval $[0, y(\varepsilon)]$ the resulting function $y_\varepsilon(\tau)$, $\tau \in R^1$, is smooth, depends smoothly on the parameters ε and u_0, and

$$\lim_{\tau\to-\infty} y_\varepsilon(\tau) = y_L = 1, \qquad \lim_{\tau\to+\infty} y_\varepsilon(\tau) = y_R = -1 + O(\varepsilon). \tag{4.7}$$

For any bounded interval $|\tau| \le T$ the grid function $y_\varepsilon(j\varepsilon)$, $|j\varepsilon| < T$, tends uniformly to the solution of the initial value problem

$$\frac{dy}{d\tau} = \frac{1}{2}(y^2 - 1), \qquad y(0) = 0 \tag{4.8}$$

as ε tends to 0. The same is obviously true for the continuous function $y_\varepsilon(\tau)$ in the interval $|\tau| < T$. This and (4.7) imply that $y_\varepsilon(\tau)$ tends uniformly in R^1 to the solution of problem (4.8), which is the same as (2.27) for $k = 1$. The trajectory $y_\varepsilon(\tau)$ in the w space is given by the equation

$$w_\varepsilon(\tau) = g(\mu y_\varepsilon(\tau), \mu, u_0) = \mu \bar{r}_0(u_0) y_\varepsilon(\tau) + O(\mu^2) \tag{4.9}$$

(compare with (3.28)), which in the u space becomes

$$u_\varepsilon(\tau) = u_0 + \mu r_0(u_0) y_\varepsilon(\tau) + O(\mu^2). \tag{4.10}$$

The eigenvalue $\lambda_0(u_\varepsilon(\tau)) = \mu y_\varepsilon(\tau) + O(\mu^2)$ so that

$$\lambda_0(u_\varepsilon(\tau))/\lambda_0(u_L) = y_\varepsilon(\tau) + O(\mu). \tag{4.11}$$

Formulas (4.10) and (4.11) imply that the trajectory $u_\varepsilon(\tau)$ satisfies the relation (2.26). It is also obvious that $\lambda_0(u_\varepsilon(\tau))/\lambda_0(u_L)$ tends uniformly on R^1 to the solution of problem (2.27). Since $y_\varepsilon(\tau)$ is decreasing, $\lambda_0(u_\varepsilon(\tau))$ is decreasing as well when $\mu > 0$ and increasing when $\mu < 0$. The case $\mu > 0$ corresponds to $\lambda_0(u_L) > 0 > \lambda_0(u_R)$; i.e., the entropy satisfying shock while $\mu < 0$ corresponds to an entropy violating shock. The local uniqueness of the discrete shock is proved by the following argument. For small μ and fixed $u_0 \in S$ let $u(\tau)$ be a discrete shock which is located in a small neighborhood of u_0 and connects $u_L(\mu, u_0)$ and $u_R(\mu, u_0)$. Then the corresponding grid function $\tilde{w}(\tau)$ satisfies the equation $H(\tilde{w}(\tau), E_\tau \tilde{w}(\tau); u_0) = 0$ and thus, by part (iii) of Theorem 5.2, belongs to the central manifold of the map H. Thus $w(\tau) = g(y(\tau), \mu, u_0)$, where the grid function $y(\tau)$ satisfies the non-scaled equation $E_\tau y(\tau) = F(y(\tau), \mu, u_0)$. For definiteness, let $\mu > 0$. Recall that $(\partial F/\partial y)(0, 0, u_0) = 1$ so that for small y and μ the function $F(y, \mu, u_0)$ is increasing with regard to the y variables. Therefore the intervals $[-\delta, y_R(\mu, u_0)]$, $[y_R(\mu, u_0), y_L(\mu, u_0)]$, and $[y_L(\mu, u_0), \delta]$ for small δ and $\mu < \delta$ are mapped by F into themselves. This implies that the above grid function $y(\tau)$ has its values only in the interval $[y_R(\mu, u_0), y_L(\mu, u_0)]$ so that in scaled variables it coincides with one of the grid functions in (4.6).

Now we consider the case $k = 3$. Recall the following result of Kopell and Howard [3] concerning the canonical problem (3.51) and (3.52).

LEMMA 4.1. (i) *There exists a solution $y(\tau)$ of the problem* (3.51) *and* (3.52) *for $k = 3$ such that $y^{(1)}(\tau)$ is an odd function and the curve $y(\tau)$ at $\tau = 0$ intersects transversally with the plane $y^{(1)} = 0$.*

(ii) *The stable (two-dimensional) manifold* $\mathfrak{M}^s$ *of the critical point* $y(\infty) = (1,0,0)^T$ *and the unstable (two-dimensional) manifold* $\mathfrak{M}^u$ *of the critical point* $y(-\infty) = (-1,0,0)^T$ *intersect transversally along the curve* $y(\tau)$.

The stable and unstable manifolds $\mathfrak{M}^s$ and $\mathfrak{M}^u$ are defined originally only in small neighborhoods of the critical points. However, one can extend them along the trajectory $y(\tau)$. We need to consider only the portions of the extended manifolds near the point $y(0)$. For any τ denote by $P(\tau)$ the solution operator of Eq. (3.51) with time increment τ and initial condition in a neighborhood of the point $y(0)$. Let $T > 0$ be large enough so that $y(T)$ belongs to the local stable manifold $\mathfrak{M}^s$ (and $y(-T)$ correspondingly to $\mathfrak{M}^u$) and let B_0 be a small ball with the center at $y(0)$ such that $P(\tau)$, when restricted to B_0, is a diffeomorphism for any $|\tau| \le T$. According to the above lemma the surfaces $\tilde{\mathfrak{M}}^s = P^{-1}(T)(\mathfrak{M}^s)$ and $\tilde{\mathfrak{M}}^u = P^{-1}(-T)(\mathfrak{M}^u)$ intersect transversally along a curve segment $\gamma = \{ y(\tau) \,|\, |\tau| < \delta \}$. We wish to repeat the above construction for the difference problem (3.47) and (3.48). Let $\mathfrak{M}^s_\varepsilon$ and $\mathfrak{M}^u_\varepsilon$ be the stable and unstable manifolds of the critical points $y_L(\varepsilon, u_0)$, $y_R(\varepsilon, u_0)$ for the map $F(y, \varepsilon, u_0)$ in (3.44). According to Theorem 5.3 these manifolds approach $\mathfrak{M}^s$ and $\mathfrak{M}^u$ in C^1 as $\varepsilon \to 0$. For $\tau \in R^1$ denote $\tau_\varepsilon = [\tau/\varepsilon]\varepsilon$ and let $P_\varepsilon(\tau_\varepsilon)$ be the map

$$P_\varepsilon(\tau_\varepsilon) = \big(F(y, \varepsilon, u_0)\big)^{[\tau/\varepsilon]} \tag{4.12}$$

restricted to $y \in B_0$. It is well known that the maps $P_\varepsilon(\tau_\varepsilon)$ approximate the maps $P(\tau)$ in C^1 uniformly in $\tau \in [-T, T]$ as $\varepsilon \to 0$. Thus the surfaces

$$\tilde{\mathfrak{M}}^s_\varepsilon = P_\varepsilon^{-1}(T_\varepsilon)(\mathfrak{M}^s_\varepsilon) \qquad \text{and} \qquad \tilde{\mathfrak{M}}^u_\varepsilon = P_\varepsilon^{-1}(-T_\varepsilon)(\mathfrak{M}^u_\varepsilon) \tag{4.13}$$

approximate the surfaces $\tilde{\mathfrak{M}}^s$ and $\tilde{\mathfrak{M}}^u$ in C^1 as $\varepsilon \to 0$. Therefore $\tilde{\mathfrak{M}}^s_\varepsilon$ and $\tilde{\mathfrak{M}}^u_\varepsilon$ intersect transversally along a curve segment γ_ε which tends to γ in C^1 as $\varepsilon \to 0$. Since γ intersects transversally with the plane $y^{(1)} = 0$ at the point $y(0)$, the curve γ_ε intersects with that plane at a unique point which we denote by $y_\varepsilon(0)$. The point $y_\varepsilon(\varepsilon) = F(y_\varepsilon(0), \varepsilon, u_0)$ for small ε also belongs to γ_ε. We parametrize the curve segment of γ_ε between the points $y_\varepsilon(0)$ and $y_\varepsilon(\varepsilon)$ by parameter $\tau \in [0, \varepsilon]$. Then for any $\tau = \tau_0 + j\varepsilon$, $\tau_0 \in [0, \varepsilon]$ and $j \in \mathbf{Z}$ we define

$$y_\varepsilon(\tau) = \big(F(y_\varepsilon(\tau_0), \varepsilon, u_0)\big)^j. \tag{4.14}$$

The parametrization in the interval $[0, \varepsilon]$ could be made in such a way that $y_\varepsilon(\tau)$ is a smooth function of τ and that $y_\varepsilon(\tau)$, $\tau \in [0, \varepsilon]$ tends in C^1 to $y(\tau)$, $\tau \in [0, \varepsilon]$ as $\varepsilon \to 0$. As a result, the function $y_\varepsilon(\tau)$ for τ in any bounded interval $|\tau| \le T$ tends uniformly in C^1 to $y(\tau)$ as $\varepsilon \to 0$. Since the function $\tau \to y(\tau)$ is one-to-one, so is the function $y_\varepsilon(\tau)$ for $|\tau| \le T$ and small ε (depending on T). Let us show that for small ε, $y_\varepsilon(\tau)$ is one-to-one, for all $\tau \in R$. Indeed, if $y_\varepsilon(\tau_1) = y_\varepsilon(\tau_2)$, without loss we can assume that $\tau_1 \in [0, \varepsilon]$ and $\tau_2 > T$. But $y_\varepsilon(\tau)$ for $\tau \ge T$ belongs to a small neighborhood of the fixed point $y_R(\varepsilon, u_0)$ which does not intersect with B_0. Since $y_R(\varepsilon, u_0) \to y(+\infty)$ and $y_L(\varepsilon, u_0) \to y(-\infty)$ it is clear that $y_\varepsilon(\tau) \to y(\tau)$ in $C(R^1)$ as $\varepsilon \to 0$. The trajectory $y_\varepsilon(\tau)$ in the w space becomes

$$w_\varepsilon(\tau) = (-1)^{(k-1)/2}\mu\left(\rho_0 y_\varepsilon^{(1)}(\tau) + \varepsilon\rho_1 y_\varepsilon^{(2)}(\tau) + \varepsilon^2\rho_2 y_\varepsilon^{(3)}(\tau)\right) + O(\mu^2), \tag{4.15}$$

and since the first n-coordinates of the vectors ρ_1 and ρ_2 are zero (see (3.14) and (3.16)), the corresponding grid function $u_\varepsilon(\tau)$ is

$$u_\varepsilon(\tau) = u_0 - \mu r_0(u_0) y_\varepsilon^{(1)}(\tau) + O(\mu^2). \tag{4.16}$$

The above relation together with the formula

$$\lambda_0(u_\varepsilon(\tau))/\lambda_0(u_L) = (-1)^{(k-1)/2} y_\varepsilon^{(1)}(\tau) + O(\mu) \tag{4.17}$$

completes the proof of parts (i) and (ii) of the Main Theorem.

Since the map $\tau \to y_\varepsilon(\tau)$ is one-to-one, in order to prove part (iv) of the theorem one should show the one-to-one correspondence between the point $y_\varepsilon(\tau)$ and the vector

$$z = z(y_\varepsilon(\tau), \varepsilon) = \left(\lambda_0(u_\varepsilon(\tau)), \lambda_0(E_\tau u_\varepsilon(\tau)), \lambda_0(E_\tau^2 u_\varepsilon(\tau))\right)^T. \tag{4.18}$$

We could as well consider $y_\varepsilon(\tau)$ in the non-scaled coordinates y. Then $u_\varepsilon(\tau)$, $E_\tau u_\varepsilon(\tau)$, and $E_\tau^2 u_\varepsilon(\tau)$ are equal correspondingly to $u_0 + w^{(0)}$, $u_0 + w^{(1)}$, $u_0 + w^{(2)}$, where $w^{(j)}$, $j = 0, 1, 2, \ldots$, are n-dimensional vector components of the vector $w = g(y, \mu, u_0)$. Using (3.28) and the definition of the matrix $X_0(u_0)$ one easily finds

$$z(y, \mu) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} y + O(|y|^2 + \mu^2), \tag{4.19}$$

so that the map $y \to z(y, \mu)$ is a diffeomorphism for small $|y|$ and μ. The theorem is proved.

Remark 4.1. The local non-uniqueness of the discrete shock in part (iv) could be caused only by a possible non-uniqueness of the solution of the problem (3.47) and (3.48) for $k = 3$. Indeed, any discrete shock u_j, $j \in \mathbf{Z}$, should lie on the central manifold $w = g(y, \mu, u_0)$, so that in the rescaled y coordinates it is a solution of the problem (3.47) and (3.48). However, the following "micro-local" uniqueness takes place: there exist constants $\delta > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, any discrete shock, which in the scaled y coordinates has a point y_j with the distance $|y_j - y(0)| < \delta$, should coincide with one of the shocks in (2.25).

5. Invariant Manifolds of an Implicit Map

In this section we prove general theorems about stable, unstable, and central manifolds used in the previous section for the proof of the Main Theorem. Let W and $\tilde{W}$ be n-dimensional smooth vector bundles over a compact manifold S. For any $p \in S$ we denote by $W(p)$ the fiber of W over p. Let $H: W \oplus W \to \tilde{W}$ be a smooth map which transforms the elements (w_1, w_2) of the fiber $W(p) \oplus W(p)$ into elements $\tilde{w} = H(w_1, w_2; p) \in \tilde{W}(p)$. We assume that the map H as well as all other non-linear maps considered below are defined only in a small neighborhood of the zero section of the corresponding bundle. Suppose that

$$H(0,0;p) \equiv 0 \tag{5.1}$$

and that the linear transformations

$$A(p) = \frac{\partial H}{\partial w_1}(0,0;p), \qquad B(p) = \frac{\partial H}{\partial w_2}(0,0;p): W(p) \to \tilde{W}(p) \tag{5.2}$$

form a regular pencil $A(p) + \lambda B(p)$ for any $p \in S$.

Theorem 5.1. *Let the bundle W be a direct sum $W = W^{cs} \oplus W^{u}$ of smooth subbundles W^{cs} and W^{u} such that for any $p \in S$ the fibers $W^{cs}(p)$ and $W^{u}(p)$ are correspondingly central-stable and unstable invariant subspaces of the pencil $A(p) + \lambda B(p)$. Then there exists a central-stable invariant manifold $\mathfrak{M}^{cs} \subset W$ which is the image of a smooth map $g: W^{cs} \to W$, and a smooth map $F: W^{cs} \to W^{cs}$, $F(0) = 0$ such that*

(i) *The compositions of g*

$$\pi^{cs} \circ g: W^{cs} \xrightarrow{g} W \xrightarrow{\pi^{cs}} W^{cs} \qquad \textit{and} \qquad \pi^{u} \circ g: W^{cs} \xrightarrow{g} W \xrightarrow{\pi^{u}} W^{u}$$

with the natural projections π^{cs} and π^{u} satisfy

$$\pi^{cs} \circ g = \text{identity}, \qquad (\pi^{u} \circ g)(0) = 0, \qquad \pi^{u} \circ \partial g(0) = 0. \tag{5.3}$$

(ii) *For any pair* $(w_1, w_2) \in W(p) \oplus W(p)$ *close to zero with* $w_2 \in \mathfrak{M}^{cs}$ *the equation* $H(w_1, w_2; p) = 0$ *is equivalent to*

$$w_1 \in \mathfrak{M}^{cs} \qquad \textit{and} \qquad w_2 = \tilde{F}^{cs}(w_1), \tag{5.4}$$

where

$$\tilde{F}^{cs} = g \circ F \circ \pi^{cs} : \mathfrak{M}^{cs} \to \mathfrak{M}^{cs}.$$

In particular, for given $g: W^{cs} \to W$ *the map* $F: W^{cs} \to W^{cs}$ *is uniquely determined by the identity*

$$H(g, g \circ F) = 0. \tag{5.5}$$

(iii) *Any sequence* $w_j \in W(p)$, $j = 0, 1, 2, \ldots$ *which is sufficiently close to zero and satisfies the equations* $H(w_j, w_{j+1}; p) = 0$, $j = 0, 1, 2, \ldots$, *belongs to* $\mathfrak{M}^{cs}$.

Proof. Since the space $W(p)$ is a direct sum of invariant (under $A(p) + \lambda B(p)$) subspaces $W^{cs}(p)$ and $W^{u}(p)$, the space $\tilde{W}(p)$ splits into a direct sum of $\tilde{W}^{cs}(p)$ and $\tilde{W}^{u}(p)$ such that the pencil $A(p) + \lambda B(p)$ could be considered as a direct sum of the restrictions

$$\begin{aligned} &A^{cs}(p) + \lambda B^{cs}(p) : W^{cs}(p) \to \tilde{W}^{cs}(p) \\ \text{and} \\ &A^{u}(p) + \lambda B^{u}(p) : W^{u}(p) \to \tilde{W}^{u}(p). \end{aligned} \tag{5.6}$$

Note also that the linear transformation

$$B^{cs}(p) \oplus A^{u}(p) : W(p) \to \tilde{W}(p) \tag{5.7}$$

is an isomorphism. We identify any vector $w = u + v \in W(p)$, $u \in W^{cs}(p)$, $v \in W^{u}(p)$ with the pair (u, v) and rewrite the equation $H(w_1, w_2; p) = 0$ as

$$H(u_1, v_1, u_2, v_2; p) = 0. \tag{5.8}$$

Then (5.7) implies that the last equation could be solved in u_2 and v_1 so that (5.8) is equivalent to

$$u_2 = H^{cs}(u_1, v_2; p), \qquad v_1 = H^{u}(u_1, v_2; p). \tag{5.9}$$

Using (5.6) one gets easily that

$$\begin{aligned} H^{cs}(u_1, v_2; p) &= M^{cs}(p) u_1 + O\big(|u_1|^2 + |v_2|^2\big), \\ H^{u}(u_1, v_2; p) &= M^{u}(p) v_2 + O\big(|u_1|^2 + |v_2|^2\big), \end{aligned} \tag{5.10}$$

where $M^{\mathrm{cs}}(p) = -(B^{\mathrm{cs}}(p))^{-1}A^{\mathrm{cs}}(p)$ and $M^{\mathrm{u}}(p) = -(A^{\mathrm{u}}(p))^{-1}B^{\mathrm{u}}(p)$, and the spectrum of $M^{\mathrm{cs}}(p)$ and $M^{\mathrm{u}}(p)$ belongs correspondingly to the sets $|\lambda| \leq 1$ and $\lambda < 1$ (compare with (3.20) and (3.21)). H^{cs} and H^{u} are actually smooth non-linear maps of the bundle W into W^{cs} and W^{u}, respectively, while $M^{\mathrm{cs}}: W^{\mathrm{cs}} \to W^{\mathrm{cs}}$ and $M^{\mathrm{u}}: W^{\mathrm{u}} \to W^{\mathrm{u}}$ are linear maps of the bundles. Identity (5.5) now becomes

$$F(u_1) = H^{\mathrm{cs}}(u_1, f(F(u_1))), \qquad f(u_1) = H^{\mathrm{u}}(u_1, f(F(u_1))). \tag{5.11}$$

(For shortness, here and below we omit the parameter $p \in S$ from the list of variables.) Here we use the notation f for the map

$$f = \pi^{\mathrm{u}} \circ g: W^{\mathrm{cs}} \to W^{\mathrm{u}}. \tag{5.12}$$

We shall solve Eq. (5.11) by iterations

$$F_{n+1}(u) = H^{\mathrm{cs}}(u, f_n(F_n(u))), \qquad f_{n+1}(u) = H^{\mathrm{u}}(u, f_n(F_n(u))), \quad u \in W^{\mathrm{cs}}(p) \tag{5.13}$$

with the first guess

$$f_0(u) \equiv 0, \qquad F_0(u) = M^{\mathrm{cs}}(p)u. \tag{5.14}$$

We can define smooth Riemann metrics on the bundles W^{cs} and W^{u} such that the maps $M^{\mathrm{cs}}(p)$ and $M^{\mathrm{u}}(p)$ have norms

$$\|M^{\mathrm{cs}}(p)\| \leq a = 1 + \varepsilon, \qquad \|M^{\mathrm{u}}(p)\| \leq b < 1 \qquad \text{for any } p \in S, \tag{5.15}$$

where $\varepsilon > 0$ is arbitrarily small (the metrics in W^{cs} depends on ε). Thus, for any given integer $r > 0$ we can assume that

$$a^r \cdot b < 1. \tag{5.16}$$

The functions $H^{\mathrm{cs}}(u, v; p)$ and $H^{\mathrm{u}}(u, v; p)$ could be changed outside of a small neighborhood of zero so that for any $p \in S$ and $u \in W^{\mathrm{cs}}(p)$, $v \in W^{\mathrm{u}}(p)$,

$$\|\partial_u H^{\mathrm{cs}}(u, v; p) - M^{\mathrm{cs}}(p)\| + \|\partial_v H^{\mathrm{cs}}(u, v; p)\| \leq \delta \tag{5.17}$$

and

$$\|\partial_v H^{\mathrm{u}}(u, v; p) - M^{\mathrm{u}}(p)\| + \|\partial_u H^{\mathrm{u}}(u, v; p)\| \leq \delta, \tag{5.18}$$

where $\delta > 0$ is arbitrarily small. Let $\alpha(\delta)$ and $\beta(\delta)$ be the smallest positive solution of the system

$$\alpha(\delta) = a + \delta + \delta\alpha(\delta)\beta(\delta), \qquad \beta(\delta) = \delta + (b + \delta)\alpha(\delta)\beta(\delta). \tag{5.19}$$

For small $\delta' = \delta/(1 - ab)$ (precisely $\delta' < 1/(2 + a + b)$) such a solution exists and has the asymptotics

$$\alpha(\delta) = a + \delta + O(\delta\delta'), \qquad \beta(\delta) = \delta' + O(\delta'^2). \tag{5.20}$$

Now we are ready to prove that

$$\sup_u \| \partial_u F_n(u) \| \le \alpha(\delta), \qquad \sup_u \| \partial_u f_n(u) \| \le \beta(\delta), \tag{5.21}$$

and for small δ'

$$f_n(u), F_n(u) \text{ are uniformly bounded in the } C^r \text{ norm.} \tag{5.22}$$

In addition

$$f_n(0) = \partial_u f_n(0) = F_n(0) = 0, \tag{5.23}$$

and

$$\| f_{n+1}(u) - f_n(u) \| \le Kc^n \|u\|, \qquad \| F_{n+1}(u) - F_n(u) \| \le K(1 - ab)c^n \|u\|, \quad c < 1. \tag{5.24}$$

As a result, the sequences $f_n(u)$ and $F_n(u)$ converge to C^{r-1} functions $f(u)$ and $F(u)$ which satisfy Eqs. (5.11) and

$$f(0) = \partial_u f(0) = F(0) = 0. \tag{5.25}$$

The proof of the above assertions is quite standard; however, we present it here with some necessary modifications which will be helpful later in the proof of Theorem 5.3. Differentiate Eqs. (5.13) and, taking into account (5.17) and (5.18), estimate supremum norms $\alpha_n = \sup_u \| \partial_u F_n(u) \|$, $\beta_n = \sup_u \| \partial_u f_n(u) \|$. We get

$$\alpha_{n+1} \le a + \delta + \delta\alpha_n\beta_n, \qquad \beta_{n+1} \le \delta + (b + \delta)\alpha_n\beta_n,$$

and (5.21) follows immediately by induction. Next, apply to (5.13) the operator ∂_u^j, $|j| \le r$ and estimate the supremum norms $\alpha_n^{(j)} = \sup_u \| \partial_u^j F_j(u) \|$, $\beta_n^{(j)} = \sup_u \| \partial_u^j f_n(u) \|$. Assuming that the norm of lower derivatives is bounded, we get

$$\alpha_{n+1}^{(j)} \leq (\delta\beta_n)\alpha_n^{(j)} + \delta(\alpha_n)^{|j|} \cdot \beta_n^{(j)} + \text{bounded terms} \tag{5.26}$$

$$\beta_{n+1}^{(j)} \leq (b+\delta)\beta_n\alpha_n^{(j)} + (b+\delta)(\alpha_n)^{|j|} \cdot \beta_n^{(j)} + \text{bounded terms.} \tag{5.27}$$

Since $\beta(\delta) < (\alpha(\delta))^{|j|}$, the sum of the above estimates provides

$$\alpha_{n+1}^{(j)} + \beta_{n+1}^{(j)} \leq (b+2\delta)(\alpha(\delta))^{|j|}\left(\alpha_n^{(j)} + \beta_n^{(j)}\right) + \text{bounded terms.} \tag{5.28}$$

The coefficient $(b+2\delta)(\alpha(\delta))^{|j|}$ for small δ' approximates $ba^{|j|} < 1$ so that the sequences $\alpha_n^{(j)}$ and $\beta_n^{(j)}$ are bounded as $n \to \infty$.

Equation (5.23) follows trivially by induction. Finally, we prove (5.24) by considering the estimates

$$\| F_{n+1}(u) - F_n(u) \| \leq \delta \| f_n(F_n(u)) - f_{n-1}(F_{n-1}(u)) \|, \tag{5.29}$$

$$\| f_{n+1}(u) - f_n(u) \| \leq (b+\delta) \| f_n(F_n(u)) - f_{n-1}(F_{n-1}(u)) \| \tag{5.30}$$

and

$$\| f_n(F_n(u)) - f_{n-1}(F_{n-1}(u)) \| \leq \beta(\delta) \| F_n(u) - F_{n-1}(u) \| + \| f_n(F_{n-1}(u)) - f_{n-1}(F_{n-1}(u)) \|. \tag{5.31}$$

Since $\|F_{n-1}(u)\| = \|F_{n-1}(u) - F_{n-1}(0)\| \leq \alpha(\delta)\|u\|$, estimates (5.24) for $n-1$ imply

$$\| f_n(F_n(u)) - f_{n-1}(F_{n-1}(u)) \| \leq Kc^{n-1}[(1-ab)\beta(\delta) + \alpha(\delta)] \|u\|. \tag{5.32}$$

Substitute the last estimate in (5.29) and (5.30), and we get estimates (5.24) for n, provided

$$(b+\delta)[(1-ab)\beta(\delta) + \alpha(\delta)] = (b+\delta)\left(a + 2\delta + O(\delta'\delta)\right) = c < 1. \tag{5.33}$$

Obviously, the limiting functions f and F satisfy (5.11) and thus (5.5). We have neglected the dependence of f and F on p. It is, indeed, easy to show that the functions $\partial_u^j f$ and $\partial_u^j F$, $|j| \leq r$, have uniformly bounded derivatives of any order (limited by the smoothness of H) with respect to $p \in S$. In order to complete the proof of part (ii) of the theorem, we should show that Eqs. (5.9) together with $v_2 = f(u_2)$ imply for small u_1, u_2, v_1, v_2 that $v_1 = f(u_1)$ and $u_2 = F(u_1)$. Substituting $v_2 = f(u_2)$ in (5.9) we get

$$u_2 = H^{\mathrm{cs}}(u_1, f(u_2); p) \tag{5.34}$$

and

$$v_1 = H^{\mathrm{u}}(u_1, f(u_2); p). \tag{5.35}$$

However, by the implicit function theorem, Eq. (5.34) for small u_1 and u_2 has a unique solution u_2, which by the identity (5.11) should coincide with $F(u_1)$. Thus $u_2 = F(u_1)$ and by (5.35) and the second identity in (5.11) we get $v_1 = f(u_1)$.

We turn now to the last part of the theorem. Represent the sequence w_j by the components $u_j \in W^{\mathrm{cs}}(p)$, $v_j \in W^{\mathrm{u}}(p)$, $j = 0, 1, 2, \dots$. Then u_j, v_j satisfy the equations

$$u_{j+1} = H^{\mathrm{cs}}(u_j, v_{j+1}; p), \qquad v_j = H^{\mathrm{u}}(u_j, v_{j+1}; p), \qquad j \geq 0. \tag{5.36}$$

Compare it with the identities

$$F(u_j) = H^{\mathrm{cs}}\big(u_j, f\big(F(u_j)\big)\big), \qquad f(u_j) = H^{\mathrm{u}}\big(u_j, f\big(F(u_j)\big)\big). \tag{5.37}$$

Define $\tilde{u}_j = u_{j+1} - F(u_j)$ and $\tilde{v}_j = v_j - f(u_j)$. The sequences $\tilde{u}_j$ and $\tilde{v}_j$ are still located in a small neighborhood of zero. Subtract equations in (5.37) from the corresponding equations in (5.36). Using (5.17), (5.18), and (5.21) we get

$$\|\tilde{u}_j\| \leq \delta \big\| v_{j+1} - f\big(F(u_j)\big) \big\| \leq \delta\big(\beta(\delta)\|\tilde{u}_{j+1}\| + \|\tilde{v}_{j+1}\|\big)$$

and

$$\|\tilde{v}_j\| \leq (b + \delta) \big\| v_{j+1} - f\big(F(u_j)\big) \big\| \leq (b + \delta)\big(\beta(\delta)\|\tilde{u}_{j+1}\| + \|\tilde{v}_{j+1}\|\big)$$

so that for small δ

$$\|\tilde{u}_j\| + \|\tilde{v}_j\| \leq (b + 2\delta)\big(\beta(\delta)\|\tilde{u}_{j+1}\| + \|\tilde{v}_{j+1}\|\big) \leq c\big(\|\tilde{u}_{j+1}\| + \|\tilde{v}_{j+1}\|\big), \qquad c < 1.$$

The last estimate implies $\tilde{u}_j = \tilde{v}_j = 0$ for all j, so that $v_j = f(u_j)$ and $w_j \in \mathfrak{M}^{\mathrm{cs}}$ for all j. The theorem is proved.

Obviously, a similar result could be proved about the central-unstable manifold if the bundle W is a direct sum

$$W = W^{s} \oplus W^{cu}, \tag{5.38}$$

such that the fibers $W^{s}(p)$ and $W^{cu}(p)$ are correspondingly stable and central-unstable invariant subspaces of $W(p)$ with regard to the pencil $A(p)+\lambda B(p)$ in (5.2). The central-unstable manifold $\mathfrak{M}^{cu}$ is an image of a map

$$g: W^{cu} \to W, \qquad \pi^{cu}\circ g = \text{identity}, \qquad \pi^{s}\circ g(0) = \pi^{s}\circ \partial g(0) = 0. \tag{5.39}$$

The corresponding map $F: W^{cu} \to W^{cu}$ satisfies, instead of (5.5), the identity

$$H(g\circ F, g) = 0. \tag{5.40}$$

Part (ii) of Theorem 5.1 now reads

(ii)′ Equation $H(w_1, w_2; p) = 0$ for small w_1, w_2 and $w_1 \in \mathfrak{M}^{cu}$ is equivalent to

$$w_2 \in \mathfrak{M}^{cu} \quad \text{and} \quad w_1 = \tilde{F}^{cu}(w_2),$$
$$\text{where } \tilde{F}^{cu} = g\circ F\circ \pi^{cu}: \mathfrak{M}^{cu} \to \mathfrak{M}^{cu}. \tag{5.41}$$

Part (iii) of Theorem 5.1 now becomes

(iii)′ Any sequence $w_j \in W(p)$, $j = 0, -1, -2, \ldots,$ which is sufficiently close to zero and satisfies the equations $H(w_j, w_{j+1}; p) = 0$, $j = 0, -1, -2, \ldots,$ belongs to $\mathfrak{M}^{cu}$.

Now we are ready to prove the central manifold theorem.

THEOREM 5.2. *Let the bundle W be a direct sum $W = W^{s}\oplus W^{c}\oplus W^{u}$ of smooth subbundles such that for any $p \in S$ the fibers $W^{s}(p)$, $W^{c}(p)$, and $W^{u}(p)$ are correspondingly stable, central, and unstable invariant subspaces of the regular pencil $A(p)+\lambda B(p)$ defined in (5.2). Then there exists a central invariant manifold $\mathfrak{M}^{c}\subset W$ which is the image of a smooth map $g: W^{c}\to W$, and a smooth map $F: W^{c}\to W^{c}$, $F(0) = 0$ such that*

(i) *The compositions of g with the natural projections $\pi^{s}: W\to W^{s}$, $\pi^{c}: W\to W^{c}$, and $\pi^{u}: W\to W^{u}$ satisfy*

$$\pi^{c}\circ g = \text{identity}, \qquad \pi^{s}\circ g(0) = \pi^{u}\circ g(0) = \pi^{s}\circ\partial g(0) = \pi^{u}\circ\partial g(0) = 0. \tag{5.42}$$

(ii) *The map F is a diffeomorphism in a small neighborhood of the zero section of W^{c}. For small $w_1, w_2 \in W(p)$, $w_1 \cap \mathfrak{M}^{cu}$, $w_2 \in \mathfrak{M}^{cs}$ the equation*

$H(w_1, w_2; p)$ is equivalent to

$$w_1, w_2 \in \mathfrak{M}^{\mathrm{c}} \tag{5.43}$$

and

$$w_2 = \tilde{F}(w_1), \qquad \text{where } \tilde{F} = g \circ F \circ \pi^{\mathrm{c}} : \mathfrak{M}^{\mathrm{c}} \to \mathfrak{M}^{\mathrm{c}}. \tag{5.44}$$

(iii) *Any sequence $w_j \in W(p)$, $j \in \mathbf{Z}$, which is sufficiently close to zero and satisfies the equations $H(w_j, w_{j+1}; p) = 0$, $j \in \mathbf{Z}$, belongs to $\mathfrak{M}^{\mathrm{c}}$.*

Proof. We define $\mathfrak{M}^{\mathrm{c}} = \mathfrak{M}^{\mathrm{cs}} \cap \mathfrak{M}^{\mathrm{cu}}$. The fibers $\mathfrak{M}^{\mathrm{cs}} \cap W(p)$ and $\mathfrak{M}^{\mathrm{cu}} \cap W(p)$ are at $w = 0$ tangential to the spaces $W^{\mathrm{cs}}(p)$ and $W^{\mathrm{cu}}(p)$. Thus $\mathfrak{M}^{\mathrm{cs}}$ and $\mathfrak{M}^{\mathrm{cu}}$ intersect transversally along the zero section of W, so that $\mathfrak{M}^{\mathrm{c}}$ is indeed a manifold near that section and $\mathfrak{M}^{\mathrm{c}} \cap W(p)$ is at $w = 0$ tangential to $W^{\mathrm{c}}(p)$. Obviously, $\mathfrak{M}^{\mathrm{c}}$ could be represented as an image of a map $g: W^{\mathrm{c}} \to W$ which satisfies conditions (5.42). Also, $\mathfrak{M}^{\mathrm{c}}$ is an invariant set of the maps $\tilde{F}^{\mathrm{cs}}$ and $\tilde{F}^{\mathrm{cu}}$ defined in (5.4) and (5.41), and

$$\tilde{F}^{\mathrm{cu}} \circ \tilde{F}^{\mathrm{cs}}|\mathfrak{M}^{\mathrm{c}} = \tilde{F}^{\mathrm{cs}} \circ \tilde{F}^{\mathrm{cu}}|\mathfrak{M}^{\mathrm{c}} = \text{identity}. \tag{5.45}$$

Indeed, if $w_1 \in \mathfrak{M}^{\mathrm{cs}} \cap \mathfrak{M}^{\mathrm{cu}}$, the element $w_2 = \tilde{F}^{\mathrm{cs}}(w_1)$ belongs to $\mathfrak{M}^{\mathrm{cs}}$ and satisfies the equation $H(w_1, w_2) = 0$. Thus by (5.41) $w_2 \in \mathfrak{M}^{\mathrm{cu}}$ and $w_1 = \tilde{F}^{\mathrm{cu}}(w_2)$. We define

$$\tilde{F} = \tilde{F}^{\mathrm{cs}}|\mathfrak{M}^{\mathrm{c}} \qquad \text{and} \qquad F = g^{-1} \circ \tilde{F} \circ (\pi^{\mathrm{c}})^{-1} : W^{\mathrm{c}} \to W^{\mathrm{c}}. \tag{5.46}$$

In view of (5.45), $\tilde{F}$ and F are local diffeomorphisms. The statements in (5.43) and (5.44) follow from (5.4) and (5.41). Part (iii) of this theorem results from parts (iii) of Theorem 5.1 and the corresponding statement (iii)′. Q.E.D.

In Sect. 4 we referred also to the following.

THEOREM 5.3. *Let $H(w_1, w_2; \varepsilon) = w_2 - w_1 - \varepsilon \hat{H}(w_1; \varepsilon)$, where $\hat{H}$ is a smooth vector function of $w_1 \in R^n$ and a real parameter ε. Assume that $\hat{H}(0; 0) = 0$ and*

$$\partial_{w_1} \hat{H}(0;0) = \begin{pmatrix} \hat{M}^{\mathrm{s}} & 0 \\ 0 & \hat{M}^{\mathrm{u}} \end{pmatrix}, \qquad \operatorname{Re} \hat{M}^{\mathrm{s}} < 0, \quad \operatorname{Re} \hat{M}^{\mathrm{u}} > 0. \tag{5.47}$$

Then the stable and unstable manifolds $\mathfrak{M}^{\mathrm{u}}_{\varepsilon}$ and $\mathfrak{M}^{\mathrm{s}}_{\varepsilon}$ corresponding to the equation $H(w_1, w_2; \varepsilon) = 0$ tend in C^r $(r \geq 1)$ to the stable and unstable manifolds $\mathfrak{M}^{\mathrm{u}}$ and $\mathfrak{M}^{\mathrm{s}}$ of the flow $dw/dt = \hat{H}(w; 0)$ as $\varepsilon \to 0$.

Proof. Equation $\hat{H}(w_1; \varepsilon) = 0$ has a zero point $w_1(\varepsilon)$ depending on ε. We can make a smooth change of variables

$$(u, v) = T(w; \varepsilon) = w + O(\varepsilon) \tag{5.48}$$

such that in the new coordinates u and v, equation $H(w_1, w_2; \varepsilon) = 0$ is equivalent to

$$\begin{aligned} u_2 &= H^s(u_1, v_2; \varepsilon) = u_1 + \varepsilon \hat{H}^s(u_1, v_2; \varepsilon), \\ v_1 &= H^u(u_1, v_2; \varepsilon) = v_2 + \varepsilon \hat{H}^u(u_1, v_2; \varepsilon) \end{aligned} \tag{5.49}$$

with

$$\hat{H}^s(0,0;\varepsilon) = \hat{H}^u(0,0;\varepsilon) = 0, \qquad \begin{pmatrix} \hat{H}^s(u,v;0) \\ \hat{H}^u(u,v;0) \end{pmatrix} = \hat{H}(u,v;0) \tag{5.50}$$

and

$$\begin{aligned} \hat{H}^s(u,v;\varepsilon) &= \hat{M}^s(\varepsilon)u + O(|u|^2 + |v|^2), \\ \hat{H}^u(u,v;\varepsilon) &= \hat{M}^u(\varepsilon)v + O(|u|^2 + |v|^2). \end{aligned} \tag{5.51}$$

Since $\hat{M}^s(\varepsilon)$ and $\hat{M}^u(\varepsilon)$ approach the matrices $\hat{M}^s$ and $\hat{M}^u$ in (5.47), for small ε

$$\operatorname{Re} \hat{M}^s(\varepsilon) \le -\delta_1 I, \qquad \operatorname{Re} \hat{M}^u(\varepsilon) \ge \delta_1 I \tag{5.52}$$

and thus the matrices $M^s(\varepsilon) = I + \varepsilon \hat{M}^s(\varepsilon)$, $M^u(\varepsilon) = I - \varepsilon \hat{M}^u(\varepsilon)$ satisfy

$$\|M^s(\varepsilon)\| \le a(\varepsilon) = 1 - \delta_2 \varepsilon, \qquad \|M^u(\varepsilon)\| \le b(\varepsilon) = 1 - \delta_2 \varepsilon. \tag{5.53}$$

(Here and below we denote by δ_i different positive constants independent of ε.) Then, in the notations of Theorem 5.1, there exists a stable manifold $\mathfrak{M}^s_\varepsilon$ which is the graph of a function $v = f(u; \varepsilon)$ and a map $u \to F(u; \varepsilon)$ such that

$$\begin{aligned} F(u;\varepsilon) &= u + \varepsilon \hat{H}^s(u, f(F(u;\varepsilon);\varepsilon);\varepsilon), \\ f(u;\varepsilon) &= f(F(u;\varepsilon);\varepsilon) - \varepsilon \hat{H}^u(u, f(F(u;\varepsilon);\varepsilon);\varepsilon). \end{aligned} \tag{5.54}$$

We shall show that the functions $f(u; \varepsilon)$ and $F(u; \varepsilon)$ are defined for $u \in \Omega$, which is some fixed neighborhood of zero, and are uniformly bounded in $C^{r+1}(\Omega)$ as $\varepsilon \to 0$. One should merely examine the proof of Theorem 5.1. We select the neighborhood Ω such that instead of (5.17) and (5.18) the estimates

$$\|\partial_u \hat{H}^s(u,v;\varepsilon) - \hat{M}^s(\varepsilon)\| + \|\partial_v \hat{H}^s(u,v;\varepsilon)\| \le \delta_3 \tag{5.55}$$

$$\|\partial_v \hat{H}^u(u,v;\varepsilon) - \hat{M}^u(\varepsilon)\| + \|\partial_u \hat{H}^u(u,v;\varepsilon)\| \le \delta_3 \tag{5.56}$$

hold. The number

$$\delta' = \delta_3\varepsilon/(1 - a(\varepsilon)b(\varepsilon)) = \delta_3/(2\delta_2) + O(\varepsilon) \tag{5.57}$$

is the very same $\delta' = \delta/(1 - ab)$ used in the proof of Theorem 5.1. Estimates (5.21) are still valid with $\delta = \delta_3\varepsilon$ and

$$\alpha(\delta) = 1 - \delta_2\varepsilon + O(\delta'\varepsilon), \qquad \beta(\delta) = \delta' + O(\delta'^2). \tag{5.58}$$

The more precise estimate for the higher derivatives of $F_n(u; \varepsilon)$ and $f_n(u; \varepsilon)$ is

$$\left\| \partial_u^j F_n(u; \varepsilon) \right\| \le K\varepsilon, \qquad \left\| \partial_u^j f_n(u; \varepsilon) \right\| \le K \qquad \text{for } |j| > 1. \tag{5.59}$$

We define $\alpha_n^{(j)}$ and $\beta_n^{(j)}$ as in Theorem 5.1. Now, however, the bounded terms in (5.26)–(5.28) are premultiplied by ε. Thus (5.28) becomes

$$\alpha_{n+1}^{(j)} + \beta_{n+1}^{(j)} \le \left(b(\varepsilon) + 2\delta\right)\left(\alpha(\delta)\right)^{|j|}\left(\alpha_n^{(j)} + \beta_n^{(j)}\right) + K\varepsilon.$$

The coefficient $(b(\varepsilon) + 2\delta)(\alpha(\delta))^{|j|} = 1 - \delta_2(|j| + 1)\varepsilon + O(\varepsilon^2) + O(\varepsilon\delta')$, which implies for small ε and δ' the uniform boundedness of $\beta_n^{(j)}$. Then estimate (5.26) gives

$$\alpha_{n+1}^{(j)} \le \delta\beta(\delta) \cdot \alpha_n^{(j)} + K\varepsilon,$$

and since by (5.14), $\alpha_0^{(j)} = 0$, we arrive at the required estimates (5.59). The convergence of the sequences $F_n(u; \varepsilon)$ and $f_n(u; \varepsilon)$ follows from the estimates (5.24) which are valid with a constant K and

$$c = \left(b(\varepsilon) + \delta\right)\left(a(\varepsilon) + 2\delta + O(\delta'\delta)\right) = 1 - 2\delta_2\varepsilon + O(\delta'\varepsilon),$$

which is less than 1 for small δ'. As a result, the limiting functions $f(u; \varepsilon)$ and $F(u; \varepsilon)$, which satisfy Eqs. (5.54), are defined for $u \in \Omega$ and are uniformly bounded in $C^{r+1}(\Omega)$ for any $r \ge 1$. Let $f(u; \varepsilon_n)$, $F(u; \varepsilon_n)$, $\epsilon_n \to 0$, be sequences which converge in $C^r(\Omega)$ to $f(u)$ and $F(u)$ correspondingly. By (5.54) we get

$$F(u; \varepsilon) = u + \varepsilon\hat{H}^{s}(u, f(u; \varepsilon); 0) + O(\varepsilon^2)$$
$$\left(f(F(u; \varepsilon); \varepsilon) - f(u; \varepsilon)\right)/\varepsilon = \hat{H}^{u}(u, f(u; \varepsilon); 0) + O(\varepsilon)$$

so that in the limit $\varepsilon_n \to 0$

$$\partial_u f(u) \cdot \hat{H}^{s}(u, f(u); 0) = \hat{H}^{u}(u, f(u); 0), \qquad f(0) = \partial_u f(0) = 0. \tag{5.60}$$

This is to say that the surface $\mathfrak{M}^s = \{(u, v); v = f(u)\}$ is the stable manifold of the flow $dw/dt = \hat{H}(w; 0)$. Indeed, for any small u_0, the trajectory $w(t)$ given by the equations $du/dt = \hat{H}^s(u(t), v(t); 0)$, $v(t) = f(u(t))$, $u(0) = u_0$ belongs to $\mathfrak{M}^s$ for all $t \geq 0$ and is also a solution of the problem $dw/dt = \hat{H}(w(t); 0)$, $w(0) = (u_0, f(u_0))$. Since the stable manifold of the flow dw/dt is unique, the limit functions $f(u)$ and $F(u)$ are independent of the sequence ε_n, so that $\lim_{\varepsilon \to 0} f(u; \varepsilon) = f(u)$, $\lim_{\varepsilon \to 0} F(u; \varepsilon) = F(u)$ in $C^r(\Omega)$. Q.E.D.

6. Appendix

Our objective here is to extend the result of Kopell and Howard [3], stated in our work as Lemma 4.1, to functions $f(y)$ more general than $y^2 - 1$ in (3.51). We consider a scalar problem

$$d^3y/dt^3 = f(y), \qquad y(-\infty) = y_L, \qquad y(+\infty) = y_R, \tag{6.1}$$

where f is a C^1 function in the interval $[-k, k]$, $-k < y_L < y_R < k$ and satisfies the conditions

$$\begin{aligned} &f(y_L) = f(y_R) = 0, \\ &f(y) < 0 \qquad \text{for } y \in (y_L, y_R), \\ &f(y) > 0 \qquad \text{for } y \notin [y_L, y_R] \end{aligned} \tag{6.2}$$

and

$$\int_{-k}^{y_R} f(y)\, dy > 0, \qquad \int_{y_L}^{k} f(y)\, dy > 0. \tag{6.3}$$

Under the above conditions, M. Mock in [6] proved that problem (6.1) has a solution. He solves an initial value problem for the equation (6.1) with the initial conditions $y(0) = p$, $y'(0) = q$, $y''(0) = 0$ and shows that for any $p \in (y_L, y_R)$ there exists $q = \sigma_p$ such that the corresponding solution $y(t)$ tends to y_R as $t \to +\infty$. Then he claims (see lines 4–16 on p. 387) that σ_p depends continuously on p and, as a result, for some p also $\lim_{t \to -\infty} y(t) = y_L$. Unfortunately, σ_p does not need to be continuous. Nevertheless, this difficulty (as shown by Mock in private communication) could be overcome if one uses the connectivity arguments for the set of all σ_p, $p \in (y_L, y_R)$. In the case of an even function $f(y)$ one can choose $p = 0$ so that the solution of the corresponding initial value problem with $q = \sigma_p$ attains also the limit y_L as $t \to -\infty$. The resulting function $y(t)$ is odd, has $y'(0) = q > 0$ and is *positive* for $t > 0$. If $f'(y) > 0$ for $y > 0$, the solution is also unique in the class of odd functions.

Now, in addition to (6.2) and (6.3) we assume

$$f(y) \text{ is even} \qquad \text{and} \qquad f'(y) > 0 \qquad \text{for } y > 0. \tag{6.4}$$

With proper scaling $y \to ay$ and $t \to bt$ we can assume that

$$y_R = -y_L = 1 \qquad \text{and} \qquad f'(y_R) = 1. \tag{6.5}$$

We rewrite problem (6.1) as a three-dimensional system

$$d\bar{y}/dt = F(\bar{y}), \tag{6.6}$$

where $\bar{y} = (y_1, y_2, y_3)^T$ and $F(\bar{y}) = (y_2, y_3, f(y_1))^T$, with conditions on infinity

$$\lim_{t \to -\infty} \bar{y}(t) = \bar{y}_L = (-1, 0, 0)^T, \qquad \lim_{t \to \infty} \bar{y}(t) = \bar{y}_R = (1, 0, 0)^T. \tag{6.7}$$

Let $\mathfrak{M}^s$ be the stable (two-dimensional) manifold of the flow in (6.6) near the critical point $\bar{y}_R$, while $\mathfrak{M}^u$ is the corresponding unstable (two-dimensional) manifold of that flow near $\bar{y}_L$. The manifolds $\mathfrak{M}^s$ and $\mathfrak{M}^u$ could be extended along the solution $\bar{y}(t)$ of problem (6.6)–(6.7) to a neighborhood of the point

$$\bar{y}_0 = \bar{y}(0) = (0, q, 0)^T, \qquad q > 0. \tag{6.8}$$

THEOREM 6. *Let the function f satisfy conditions* (6.2)–(6.5). *Then the stable and unstable manifolds of the flow in* (6.6), *corresponding to the critical points $\bar{y}_L$ and $\bar{y}_R$, respectively, intersect transversally along the trajectory $\bar{y}(t)$.*

Proof. Denote by $T\mathfrak{M}^s(\bar{y})$ and $T\mathfrak{M}^u(\bar{y})$ the planes tangent to the manifolds $\mathfrak{M}^s$ and $\mathfrak{M}^u$ at the point $\bar{y}$. The planes $T\mathfrak{M}^s(\bar{y}_0)$ and $T\mathfrak{M}^u(\bar{y}_0)$ have a common vector $[d\bar{y}(t)/dt]_{t=0} = (q, 0, f(0))^T$. Suppose that $T\mathfrak{M}^s(\bar{y}_0)$ and $T\mathfrak{M}^u(\bar{y}_0)$ are not transversal; i.e., they coincide. Note that problem (6.6)–(6.7) is invariant under the transformation

$$\bar{y} \to \Phi(\bar{y}) = (-y_1, y_2, -y_3)^T, \qquad t \to -t$$

so that Φ transforms $\mathfrak{M}^s$ into $\mathfrak{M}^u$ and vice versa. Therefore the plane $T\mathfrak{M}^s(\bar{y}_0) = T\mathfrak{M}^u(\bar{y}_0)$ is invariant under the transformation Φ. This leaves for $T\mathfrak{M}^s(\bar{y}_0)$ only two possibilities:

$$T\mathfrak{M}^s(\bar{y}_0) = \mathrm{Sp}\{(1, 0, 0)^T, (0, 0, 1)^T\}$$

or

$$T\mathfrak{M}^s(\bar{y}_0) = \mathrm{Sp}\{(q, 0, f(0))^T, (0, 1, 0)^T\}. \tag{6.9}$$

Denote by R^3_+ the positive octant $\bar{y} \neq 0$, $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$. We shall show that $T\mathfrak{M}^s(\bar{y}_0)$ does not intersect with R^3_+. For that sake we linearize Eq. (6.6) near the solution $\bar{y}(t)$ and consider the initial value problem

$$d\bar{z}/dt = A(t)\bar{z}, \qquad t > 0, \quad \bar{z}(0) \in R^3_+, \tag{6.10}$$

where

$$A(t) = \partial_{\bar{y}} F(\bar{y}(t)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f'(y_1(t)) & 0 & 0 \end{pmatrix}.$$

Since all coefficients of $A(t)$ are positive for $t > 0$, the solution of problem (6.10) belongs to R^3_+ for all $t \geq 0$. The tangent plane $T\mathfrak{M}^s(\bar{y}_R)$ coincides with the stable invariant space of the orthogonal matrix

$$A(+\infty) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

so that $T\mathfrak{M}^s(\bar{y}_R)$ is orthogonal to the "unstable" direction $(1,1,1)^T$. This implies that $T\mathfrak{M}^s(\bar{y}_R)$ does not intersect with R^3_+, and by continuity it is correct also for $T\mathfrak{M}^s(\bar{y}(t_0))$ if t_0 is large enough. Let $P(t_0)$ be the solution operator for Eq. (6.6) with initial condition near $\bar{y}_0$ and time increment t_0, and let $dP(t_0)[\bar{y}_0]$ be the differential of $P(t_0)$ at $\bar{y}_0$. Obviously $dP(t_0)[\bar{y}_0]$ transforms the vector $\bar{z}(0)$ into $\bar{z}(t_0)$ and also the tangent space $T\mathfrak{M}^s(\bar{y}_0)$ into $T\mathfrak{M}^s(\bar{y}(t_0))$. Since $dP(t_0)[\bar{y}_0]$ is an isomorphism and $\bar{z}(t_0) \notin T\mathfrak{M}^s(\bar{y}(t_0))$, we get $\bar{z}(0) \notin T\mathfrak{M}^s(\bar{y}_0)$ so that $R^3_+ \cap T\mathfrak{M}^s(\bar{y}_0) = \varnothing$. Obviously, both cases exhibited in (6.9) do not satisfy the last condition. We have proved that the manifolds $\mathfrak{M}^s$ and $\mathfrak{M}^u$ intersect transversally at $\bar{y}(0)$ and hence at any $\bar{y}(t)$.

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